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Infinite-Dimensional Groups and Borel Structures with Subalgebras of Ultraproduct

الزمر لانهائية البعد وبناءات بورل مع الجبريات الجزئية للمنتج الفوقي

A Thesis Submitted in Fulfillment of the Requirement for the Degree of M.Sc. in Mathematics

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Dedication

To my Family.

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Abstract

We show the imprimitivity and induced representations of locally compact groups I. We study the Kadec norms and Borel sets in Banach spaces and function spaces with the weak topology. We investigate the problem of Kadison on maximal abelian and injective subalgebras in factors associated with free-groups. We obtain quasi-regular and induced representations of the infinite-dimensional nilpotent groups. We discuss problems concerning Borel structures in function spaces and in the Banach spaces with Baire measurability in spaces of continuous functions. We give the independence properties in subalgebras of ultraproduct II_1 factors and factors of type II_1 without non-trivial finite index subfactors.

الخلاصة

أوضحنا التمثيلات غير البدائية والمحدثة لزمر التراص الموضعية I. درسنا نظائم كاديك وفئات بورل في فضاءات باناخ و في فضاءات الدالة مع الطبولوجيا الضعيفة. لقد درسنا مسألة كاديسون على الأبيلية العظمى والجبريات الجزئية الأحادية في العوامل المشاركة مع زمر-الحرة. تحصلنا على التمثيلات شبه المنتظمة والمحدثة للزمر ذات القوى الصحيحة الموجبة المساوية للصفر لانهائية-البعد والزمر. ناقشنا بعض المسائل المختصة ببناءات بورل في فضاءات الدالة وفضاءات باناخ مع مقييس بايير في فضاءات للدوال المستمرة. أعطينا الخصائص على المستقلة في الجبريات الجزئية للناتج الفوقي لعوامل II_1 وعوامل النوع II_1 بدون العوامل الجزئية للدليل المنتهي غير البدائي.

Introduction

We shall discuss a generalization of this notion which is more suitable for use in connection with infinite dimensional representation because it allows the direct sum decomposition to be continuous as well as discrete.

This connection between representations of groups and representation of their subgroups has many interesting and useful properties in the finite case and it naturally occurs to on to study the extent to which these properties persist in general. We introduce a property for a couple of topologies that allows us to give simple proofs of some classic results about Borel sets in Banach spaces by Edgar, Schacbermayer and Talagrand as well as some new results. It is show that the duality map$\langle\cdot\rangle: (\ell^{\infty}, weak) \times ((\ell^{\infty})^{*}, weak^{*}) \to R$ is not Borel. More generally, the evaluation $e: (C(K), weak) \times K \to R, e(f, X) = f(x)$, is not Borel for any function space C(K) on a compact F-space.

We show that under certain conditions Kadison's problem has an affirmative answer. We also show by a counter example that the hypothesis of separability is essential a von Neuman algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is called injective. A von Neuman algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is called injective if there exists a norm one projection from the Banach algebra of all linear bounded operators on $\mathcal{H}$ onto $\mathcal{A}$. As the injective von Neumann algebras form a monotone class, any von Neumann algebra has maximal injective von Neumann subalgebras.

In the present work an analog of the quasiregular representation which is well known for locally-compact groups is constructed for the nilpotent infinite-dimensional group $B_0^{\mathbb{N}}$ and a criterion for its irreducibility is presented. The induced representation $\mathrm{Ind}_H^G S$ of a locally compact group G is the unitary representation of the group G associated with unitary representation S : H → U(V) of a subgroup H of the group G. To develop the concept of in- duced representations for infinite-dimensional groups. The induced representations for infinite-dimensional groups in not unique, as in the case of a locally compact groups.

It is an open problem if any separable compact space *K* whose function space *C(K)* with thecylindrical σ-algebra is a standard measurable space, embeds in the space of the first Baire class functions on the Cantor set, with the pointwise topology. We prove that this is true for separable linearly orderedcompacta.

Let C(K) be the Banach space of all continuous functions on a given compact space K. We investigate the w^{*}-sequential closure in $C(K)^{*}$of the set ofall finitely supported probabilities on K. M. Talagrand showed that, for the ˇCech-Stone compactification $\beta\omega$ of the space of natural numbers ω , the norm and the weak topology generate different Borel structures in the Banach space C ($\beta\omega$).

We call a sub factor N $\subset$ M trivial if it is isomorphic with the obvious inclusion of N in $M_2(\mathbb{C}) \otimes N$. We prove the existence of type II_1 factors M without non-trivial finite index sub factors. Equivalently, every M-M-bimodule with finite coupling constant, both as a left and as a right M-module, is a multiple of $L^2(M)$. We show that if $Q \subset M$ is either an ultraproduct $Q = \Pi_\omega Q_n$ of subalgebras $Q_n \subset M_n$ with $Q_n \nprec M_n\, Q'_n \cap M_n \forall n$, or the centralizer $Q = B' \cap M$ of aseparable amenable $*$-subalgebra $B \subset M$ then for any separable subspace $X \subset M \ominus (B' \cap M)$, there exists a diffuse abelian von Neumann subalgebra in Q which is free independent to X , relative to $Q' \cap M$.

The Contents

Chapter 1
Locally Compact Groups

We study the imprimitivity and representations of the locally compact groups1and ergodicity and transitivity. We show ad determine the induced representations of locally compact groups 1.

Section (1.1): Imprimitivity for Representations

In the classical theory of representations of finite groups by linear transformations a representation $s \rightarrow U_s$ of a finite group is said to be imprimitive if the vector space H in which the U_s act is a direct sum of independent subspaces $M_1, M_2, \ldots, M_n$, in such a manner that each U_s transforms each M_{i} into some M_{j}. In the present note we shall discuss a generalization of this notion which is more suitable for use in connection with infinite dimensional representation because it allows the direct sum decomposition to be continuous as well as discrete. Our principal theorem (well known for finite groups) deals with weakly (and hence strongly) continuous unitary representations of separable locally compact groups. It asserts that the pair consisting of such a representation and a "transitive system of imprimitivity" for it defines an essentially unique subgroup G_0 and an essentially unique representation L of G_0 from which the original pair may be reconstructed in a quite explicit manner.

This result has a number of applications. A recent theorem [2] which implies the Stone-von Neumann theorem on the uniqueness of operators satisfying the Heisenberg commutation relations is included as a special case. In addition it may be used to give a complete determination of the irreducible unitary representations of the members of a class of locally compact groups which are neither compact nor Abelian.

Definition (1.1.1)[1]: Let $s \rightarrow U_s; -M_1, M_2, \ldots, M_n$, Be an imprimitive representation in the classical sense. Suppose that the U, are unitary and that the M_{i} are mutually orthogonal. Let M denote the set of integers 1, 2, …,n. For each s in the group G and each $j \in M$ let $(j)s$ be the index of the subspace into which U_{s-1} carries $M_j. Let\ P_j$ denote the projection of H on M_{j}. Then it is easy to see that $U_s P_j U_s^{-1} = P_{(j)s-1}$. More generally if P_{E} is defined by the equation $P_E = \sum_{(j \in E)} P_J$ for each $E \subseteq M$ then $U_s P_E U_s^{-1} = P_{(E)s^{-1}}$. The motivation for the following definition should now be clear. Let M be a separable locally compact space and let G be a separable locally compact group. Let $x, s \rightarrow (x)s$ denote a mapping of $M \times G$ onto M which is continuous and is such that (a) for fixed $s, x \rightarrow (x)s$ is a homeomorphism and (b) the resulting map of G into the group of homeomorphisms of M is a homomorphism.

Let $P(E \rightarrow P_E)$ be a σ homomorphism of the σ Boolean algebra of all Borel subsets of M into a σ- Boolean algebra of projections in a separable Hilbert space H such that$P_M$is the identity I. Let $U(s \rightarrow U_s)$ be a representation of G in H; that is a weakly (and hence strongly) continuous homomorphism of G into the group of unitary operators in H. If $U_S P_E U_s^{-1} = P_{(E)S^{-1}}$ for all E and s and if P_E takes on values other than 0 and I we shall say that U is imprimitive and that P is a system of imprimitivity for U. We shall call Il the base of P. It is to be observed that P defines in Ma family of null sets and that there exists in M a family of mutually equivalent measures whose sets of measure zero are precisely these null sets.

The null sets are those sets E for which $P_E = 0$ and the measures are those of the form $\mu(E) = (P_E f, f)$ where f is an element [3] in H such that $P_E f = 0$ implies $P_E = 0$.

Ergodicity and Transitivity.-When for each x and y in M there exists s in G for which $(x)s = y$ it is natural to say that P is a transitive system of imprimitivity for U. When M is finite every system of imprimitivity decomposes in a natural manner into transitive ones corresponding to the orbits of M under G. The decomposition of M into orbits is not reflected in a corresponding decomposition of H. It is rather the decomposition of M into ergodic or metrically transitive parts which is relevant. We define a system of imprimitivity P to be ergodic if G acts ergodically on the base M of P; that is, whenever $(E)s$ differs from E by a null set for all $s \in G$ then E is itself a null set or the complement of one. In view of the current literature on the decomposition of measures the study of general systems of imprimitivity may be expected to be reducible to the study of ergodic systems.

Ergodic systems which are not also transitive are rather difficult to handle and such results as we have at present are far from definitive. We deal exclusively with transitive systems. Fortunately in some applications it can be shown that only transitive systems can arise. Specifically let us say that the orbits of M under G are regular if there exists a countable family $E_1, E_2, \ldots$ of Borel subsets of M, each a union of orbits such that each orbit of M is the intersection of the members of a sub-family $E_{n_1}, E_{n_2}, \ldots$. Then the following theorem is easily showd.

Theorem (1.1.2)[1]: If the orbits of M under G are regular then for each ergodic system of imprimitivity based on M there is an orbit C such that $P_{M-C} = 0$.

Formulation of the Principal Theorem.-Let P be a transitive system of imprimitivity for the representation U of the separable locally compact group G. Let x_0 be a point of the base M of P. Let G_0 be the set of all $s \in G$ for which $(x_0)s = x_0$. Then G_0 is a closed subgroup of G and the mapping $s \to (x_0)s$ of G on M defines a one-to-one Borel set preserving map of the homogeneous space G/G_0 of right G_0 cosets onto M. Thus P is equivalent in an obvious sense to another system of imprimitivity for U whose base is the homogeneous space G/G_0. In general we shall define a pair to be a unitary representation for the group G together with a particular system of imprimitivity for this representation. If U, P and U', P' are two pairs with the same base M we shall say that they are unitary equivalent if there exists a unitary transformation V from the space of U and P to the space of $U' \ and \ P'$ such that $V^{-1}U'_sV = U_s$ and $V^{-1}P'_EV = P_E$ for all s and E. It follows from the above remarks that the problem of determining to within unitary equivalence all pairs based on a given M may always be reduced to the corresponding problem in which M is a homogeneous space. We shall accordingly confine ourselves to this case. The arbitrariness in the choice of x_0 has the effect only of providing us with several essentially equivalent complete systems of invariants for the pairs based on a given M.

We describe a method (which will show to be general) of constructing pairs based on a given G/G_0. Let, μ be a finite Borel measure on G/G_0 which is "quasi invariant" in the sense that the action of G on G/G_0preserves null sets.[4] Let $L(\xi \to L_\xi)$ be a representation of G_0 by unitary operators in a Hilbert space H_0. Then let H_{L} be the set of all functions f from G to H_0 such that: (a) f is a Borel function in the sense that $(f(s), v)$ is a Borel function of s for all $v \in H_0$;

(b) for all $s \in G$ and all $\epsilon \leftarrow G_0, f(\xi s) = L_\xi f(s)$; and (c) $(f(s), f(s))$ (which by (b) is constant on the right G_0 cosets) defines a summable function on G/G_0 . By a more or less obvious adaptation of the proof of the Riesz Fischer [6] it may be shown that H_L is a Hilbert space with respect to the inner product $(f, g)_L = \int_{G/G}(f(s), g(s))d\mu$, and the obvious linear operations. Naturally functions which are equal almost everywhere are to be identified. Now let P be the function on $G \times G/G_0$ which for each fixed s is the Radon Nikodym derivative of the translate of μ $by\ s$ with respect to μ itself. Then regarding ρ , as we may, as a function on $G \times G$ let $U_s f$ for all $s \in G$ and $f \in H_L$ be defined by the equation $(U_s f)(t) = f(ts)/\sqrt{\rho(s^-, ts)}$. It is readily verified that U_s, is a unitary transformation of H_L onto itself and that the mapping $s \to U_s$ is a representation of G. For each Borel subset E of $G / G_0\ let\ \emptyset$ be its characteristic function regarded as a function on G . For $f \in H_L\ let\ (P_E f)(t) = \emptyset(t)f(t)$. It is easy to see that the mapping $f \to P_E f$ is a projection and that U and P together constitute a pair in the sense of the above definition. We shall call it the pair generated by $and\mu$. We can now formulate our main theorem.

Theorem (1.1.3)[1]: Let G be a separable locally compact group and let G_o be a closed subgroup of G. Let U', P' be any pair based on G / G_0. Let μ be any quasi invariant measure in G / G_0.. Then there exists a representation L of G_0 such that U', P' is unitary equivalent to the pair generated by L and A. If L and L' are representations of G_0 and, μ and μ' are quasi invariant measures in G / G_0.then the pair generated by L' and μ' is unitary equivalent to the pair generated by L and, u if and only if L and L' are unitary equivalent representations of G_0.

Proof. We shall give the proof in outline only leaving relatively routine details to the reader. Moreover we shall assume familiarity on the part of the reader with the section cited in [2] and will omit arguments similar to those given there. We shall refer to this section as SVN. The proof falls naturally into two parts. First we show that every pair defines a representation of G_0 unique to within unitary equivalence and that two pairs defining equivalent representations of G_0 are unitary equivalent. Then we complete the proof by showing that the representation of G_0 defined by the pair generated by an arbitrary L and μ is unitary equivalent to L itself.

Given a pair U', P' based on G / G_0. we note first that the set of all $P_{E'}$ is a uniformly n dimensional Boolean algebra of projections $(n = 1, 2, \cdots, \infty)$ in the sense of Nakano (see SVN 5). This follows from the fairly easily showd fact that G acting on G / G_0 is ergodic. Let N denote an n dimensional identity representation of $G_0, let\ \mu$ be a quasi invariant measure in G / G_0.and let W,P be the pair generated by N and IA. Just as in No. 6 of SVN it is possible to show that the pair U', P' is unitary equivalent to the pair U,P where P comes from the pair W,P above and U is a suitable representation of G. We define $Q_s\ as\ U_s W_s^{-1}$ and observe that $Q_S P_E = P_E Q_s$ for all E and s. It follows as in SVN that there exists a weakly Borel function $Q^\sim$ from G× to the group of unitary operators in the space H_1in which the N_s, act such that for each s in G we have $(Q_s f)(t) = Q^\sim(s, t)f(t)$. The identity $Q^\sim(s_1 s_2, t) = Q^\sim(s_1, t)Q^\sim(s_2, ts_1)$ holding for almost all triples is established as in SVN and from it the existence of a weakly Borel function B such that $Q^\sim(s, t) = B^{-1}(t)B(ts)$ almost everywhere. The fact that the functions in H_N are constant on the right G_o cosets implies that $Q^\sim(s, \xi t) = Q^\sim(s, t)$ for all $\xi \in G_0$ almost everywhere in s and t. This implies in turn that $B^{-1}(\xi t)B(\xi ts) = B^{-1}(t)B(ts)$ in the same sense or equivalently that $B(\xi ts)B^{-1}(ts) =$

$B(\xi t)B^{-1}(t)$. In short for each $\xi \in G_0, B(\xi t)B^{-1}(t)$ is almost everywhere equal to a certain constant operator L_ξ . A simple argument shows that $(L_\xi v_0, v_1)$ is of the form $\int_G \psi(t)\ (B(\xi t)v_2, v_1)$ for a dense set of $v_0's$. Here v_0, v_1 and v_2 are elements in H_1 and $\psi$is a continuous complex valued function vanishing outside of a compact subset of G. It follows readily that $(L_\xi v_0, v_1,)$ is continuous in ξ and, since $L_{\xi 1 \xi 2} = L_{\xi 1}\ L_{\xi 2}, that\ L(\xi \to L_\xi)$ is a representation of G_0. Of course L may depend upon the choice of μ, the choice of the unitary map of the given Hilbert space on H_N and the choice of B. However, the fact that any two $\mu's$ have the same null sets guarantees the lack of dependence of $L\ 0n\ \mu$. As to the other possible dependencies note that a unitary map $X\ 0f\ H_N$ on itself which commutes with all P_E is defined by an equation of the form $Xf(t) = X(t)f(t)\ where\ X(t)$ is a unitary operator on H_1 for each t and $X(t)$ is a weakly Borel function of t. Moreover $X(\xi t) = X(t)$ for $\xi \in G_0$. It is readily calculated that the effect on Q of a transformation by X is to replace it by R where $R(s,t) = X^{-1}(t)Q(s,t)X(ts)$. Now if $C^{-I}(t)C(ts) = X^{-1}(t)B^{-I}(t)B(ts)X(ts)$ it follows that $B(t)X(t)C^{-1}(t)$ is (modulo null sets) independent of t. Thus for some constant operator K we have $C(t) = KB(t)X(t)$ so that $C(\xi t)C^{-1}(t) = KB(\xi t)X(\xi t)X^{-1}(t)B^{-I}(t)K^{-1} = KL_\xi K^{-1}$. In short our original pair and in fact the unitary equivalence class to which it belongs determine L to within unitary equivalence.
Conversely a simple reversal of the argument shows that pairs leading to unitary equivalent $L's$ must be themselves unitary equivalent.

Now let L' be an arbitrary representation of G_0 and let U', P' be the pair generated by L' and a quasi invariant measure $\mu\ in\ G/G_0$. By the argument of the preceding paragraph there is a unitary map V^{-1} of $H_L,$ on some H_N such that $V^{-1}P_E,V = P_E$ where W, P is as before the pair generated by N and μ and N is an identity representation of G_0 on a Hilbert space H_1. It is not difficult to show that there exists a weakly Borel function $V^\sim$ defined on G whose values are operators from H_1 to the space H_2 in which L' operates such that $(Vf)(t) = V^\sim(t)f(t)$. It follows from the fact that V is unitary that $V^\sim(t)$ is unitary from $H_1\ int0\ H_2$ for almost all t and it follows from the fact that $Vf \in H_L,$ that for each $\xi \in G_0, V^\sim(\xi t) = L_\xi, V^\sim(t)$ for almost all t. Now the Q_s of the preceding paragraph here take the form $V^{-1}\ U'_s V W_s^{-1}$ so that $U'_s V W_s^{-1} = VQ_s$. Hence $V^\sim(ts) = V^\sim(t)Q^\sim(s,t)\ 0r\ V^\sim(ts) = V^\sim(t)B^{-1}(t)B(ts)$ or $V^\sim(ts)B^{-1}(ts) = V^\sim(t) - B^{-1}(t)$.

Thus there exists a norm preserving operator K independent of t such that $V^\sim(t) = KB(t)$ for almost all t. If K were known to map H_1 onto the whole of H_2 we could write $B(t) = K^{-1}V^\sim(t)$ and conclude at once that $B(\xi t)B^{-1}(t) = K^{-1}L_{E'}\ V^\sim(t)V^\sim - 1'(t)K = K^{-1}L_\xi K$ and hence that the $L\ f0r\ U', P'$ is unitary equivalent to L'. In order to show that K is indeed an onto mapping we must make use of certain facts about the space $H_{L'}$ which so far as we know at this point might be zero dimensional. For each continuous function w from $G\ t0\ H_2$ which vanishes outside of a compact subset of G let $\overline{w}$ be defined by the equation $(\overline{w}(t), v) = \int_{G_0} (L_{\xi'^{-1}} w(\xi t), v) dt\xi$ for all v in H_1 and all t in G. This function may be shown to be a continuous member of $H_{L'}$ which vanishes outside of a set whose image in GIG_0 is compact. Arguments of a fairly routine nature show that for each t e G the vectors $\overline{w}\ (t)$ span H_2. Now suppose that K does not map $H_1\ 0nt0\ H_2$. Choose v_0 orthogonal to the range of K. Consider an arbitrary member of H_L, of the form $\bar{w}$. We have $\bar{w}(t) = V^{\sim}(t)f(t)$ for some f and almost all t. But $V^{\sim}(t)f(t)$ is in the range of K for almost all t.

Thus, since $\bar{w}$ is continuous we can conclude that $(\bar{w}\,(t), v_0) = 0$ for all t. Hence for all $t, (\bar{w}(t), v_0) = 0$ for all $\bar{w}$ and this contradicts the fact that the $\bar{w}(t)$ span for each t.

The natural question concerning the connection between the reducibility of a pair U, P and the reducibility of the defining representation of G_0 is easily answered. If T commutes with all L_ξ then a transformation $T^{\sim}$ taking H_N into H_N is defined by the equation $(T^{\sim}f)\,(t) = B^{-1}(t)TB(t)f(t)$ where B is the function used in defining L. Then, as is easily seen $T \to T^{\sim}$ is a $*-$isomorphism of the ring of all bounded linear operators which commute with all the L_ξ onto the ring of all bounded linear operators which commute with all the U_S and all of the P_E. In particular the U_s and the P_E are simultaneously reducible if and only if L is a reducible representation of G_o.

Application to the Determination of Group Representations.-Let G be a separable locally compact group and let G_1 be a closed normal Abelian subgroup of G. Let $\hat{G}_1$ denote the character group of G_1. Every member s of G defines an automorphism $x \to sxs^{-1}$ $0f$ G_1 and this in turn induces an automorphism $y \to (y)s$ $0f$ $\hat{G}_1$. Now let U be any irreducible representation of G. Restricted to G_1 it admits a spectral resolution defined by a a homomorphism P of the Borel subsets of $\hat{G}_1$, into a Boolean algebra of projections in the Hilbert space H in which U acts. An obvious calculation shows that $U_S P_E U_S^{-1} = P_{(E)S-1}$. Thus P is a system of imprimitivity for U. Since U is irreducible P must be ergodic. If we assume that G_1 is "regularly imbedded" in G in the sense that the orbits in G_1 under G are regular then Theorem (1.1.2) tells us that G_1 may be replaced by a single orbit. Let y be a point in this orbit and let G_v be.the closed subgroup of all s for which $(y)s = y$. Theorem (1.1.3) tells us that U is unitary equivalent to the first member of the pair generated by an irreducible representation of G_v.

If G is a "semi-direct product" of G_1 and GIG_1; that is, if there exists a closed subgroup G_2such that $G_1 n\, G_2 = e$ and $G_1G_2 = G$ much more precise information is available.

Theorem (1.1.4)[1]: Let G_1 be imbedded regularly in G and let G be a semidirect product of G_1 and G_2. From each orbit C of G_1 under G_2 choose a member Yc. Let G_C, denote the set of all $s \in G_2$ with $(yc)s = yc$. Then the general irreducible representation of G may be obtained as follows. Select an orbit C and an irreducible representation L of G_C. Let M be the irreducible representation of $G_1 - G_C$ which coincides with L on G_C and is y_C times the identity on G_1. Then the first member of the pair generated by Mand a quasi invariant measure in $G/(G_1.G_C)$ is the required irreducible representation of G.

Every irreducible representation of G may be so obtained and two such are unitary equivalent if and only if they come from the same orbit and unitary equivalent $L's$.

When the irreducible representations of G_2 and its subgroups are known Theorem (1.1.4) furnishes a complete description of the irreducible representations of G. This is so, in particular, when G_2 is Abelian. Moreover when G_2 is Abelian (and G_1 is imbedded regularly in G) it tells us that every

Irreducible representation of G is of "multiplier" form. More generally any imprimitive representation of G generated by a one-dimensional representation of a subgroup is unitary equivalent to a representation in which the underlying Hilbert space is the space of square summable functions on a homogeneous space and the action of the operator associated with s is to translate by s and multiply by a certain function (the multiplier) of s and a variable point in the homogeneous space.

When G_1 is not imbedded regularly in G Theorem (1.1.4) fails only in that it does not describe all of the irreducible representations. The ones that it does describe still exist and are irreducible. We have examples, however, showing that in general there are many others. Their existence leads to various kinds of pathological behavior which we expect to discuss at another time. Since these "extra" representations are all infinite dimensional, Theorem(1.1.4) provides an analysis of all finite dimensional representations for arbitrary semidirect products.

A number of well-known groups are regular semidirect products and Theorem (1.1.4) includes as special cases results in the literature analyzing their representations. Examples include the unique non-commutative two-parameter Lie group [6] (a semidirect product of two lines) and the group of Euclidean motions in two space [7] (a semidirect product of the two-dimensional translation group and the circle group). Wigner's [7] reduction of the representation problem for the inhomogeneous Lorentz group to that for the homogeneous Lorentz group is also a consequence of Theorem (1.1.4) since the former group is a semidirect product of a translation group and the latter group.

(i) In the mapping from a representation L of G_0 to the pair it generates one can ignore P and obtain a mapping from representations of G_0 to representations of G. It is not difficult to see that this mapping carries the regular representation of G_0 into the regular representation of G. Thus (in view of No. 6) any analysis of the regular representation of G_0 as a direct sum or integral will define a corresponding analysis of the regular representation of G although the "parts" will not necessarily be irreducible. This decomposition when G_0 is Abelian is the subject of a recent interesting note of G_0 dement [8]. It was this note of G_0 dement together with a discussion of such a space for compact groups given by A. Weil [9] that suggested our definition of the Hilbert space H_L.

There are a number of questions suggested by the considerations, which we expect to investigate and report. We close by mentioning a few of these. (i) When is the representation U of G generated by an irreducible representation L of G_0 itself irreducible?

(ii) When G is finite, L and U are finite dimensional and L is irreducible there is a classical theorem which says that the number of times that U contains a given irreducible representation V of G is equal to the number of times that the restriction of V to G_0 contains L. Weil [9] has recently extended this theorem to compact groups. One can ask whether (and in what sense) it continues to be .true for general locally compact groups.

(iii) To what extent is it true that an arbitrary irreducible representation of G is the imprimitive representation generated by a primitive representation L of an appropriate G_0? How is the possible failure of this to hold generally connected with the "extra" representations of non-regular semidirect products? (iv) Theorem (1.1.3) presumably can be used to show other theorems like Theorem (1.1.4). What are some of these? One notes in particular that G_1 can probably be replaced by any group whose representations can be decomposed into irreducible parts in a suitably manageable manner.

Section (1.2): Induced Representations

In the theory of representations of a finite group by linear transformations the closely related notions of "imprimitivity" and of "induced representation" play a prominent role. [28] has generalized these notions to the case in which the group is a Separable locally compact topological group and the linear transformations are unitary transformations in Hilbert space. It turns out (and this is the principle Theorem of [28]) that the classical Theorem of Frobenius according to which every "imprimitive" representation of a finite group is "induced" in a certain canonical fashion by a representation of a subgroup may be reformulated so as to remain true under the more general circumstances indicated above. This connection between representations of groups and representations of their subgroups has many interesting and useful properties in the finite case and it naturally occurs to one to study the extent to which these properties persist in general.

The principal results, formulated as Theorems (1.2.34), (1.2.35) and (1.2.37) , are closely related; each being essentially a Corollary of its predecessor. The first asserts that if L is a representation of the closed subgroup G_1 of $\mathfrak{G}$ and U^L is the corresponding induced representation of $\mathfrak{G}$ then the restriction of U^L to the closed subgroup G_2 is a "sum" over the G_1: G_2 double cosets of certain induced representations o G_2.
The second gives a similar decomposition of the Kronecker product U^L f$\otimes$ U^M where L and M are representations of G_1 and G_2 respectively.

The third provides a usable formula for computing the "strong intertwining numbers" of the induced representations U^L and U^M of $\mathfrak{G}$. The "sums" in question are ordinary discrete sums only when there are at most countably many non trivial double cosets.
In general "direct integrals" as defined by von Neumann in [36] must be used and we must restrict ourselves to the case in which the relevant double coset decomposition of $\mathfrak{G}$ is "measurable".

As we have shown in detail else where [30] these Theorems for finite groups imply certain classical results; in particular the Frobenius reciprocity Theorem and the shoda criteria for the irreducibility and unitary equivalence of monomial representations. The Theorems yield generalizations of these results but these generalizations may be regarded as satisfactory only insofar as they deal with representations whose irreducible constituents are discrete and finite dimensional.

We have made a start in [30] and hope to be able to discuss the situation more fully in [19], [28].

Let $\mathfrak{E}$ be a separable locally compact group and let G be a closed subgroup of $\mathfrak{E}$. Let $\mathfrak{M}$ be the homogeneous space of right G cosets and let h (x $\longrightarrow$ h(x))denot the canonical mapping of $\mathfrak{G}$ onto $\mathfrak{M}$. If $\mathrm{z} = \mathrm{h}(\mathrm{x}) \in \mathfrak{W}$ and $\mathrm{y} \in \mathfrak{E}$ then h(xy) depends only upon y and $h(x)$.

We shall denote this element by [z]y. It is readily verified that z → [z]y is a homeomorphism, that $[\mathrm{z}]\mathrm{y}_\mathrm{I}\mathrm{y}_2 = [[\mathrm{z}]\mathrm{y}_1]\mathrm{y}_2$ and that $[\mathrm{z}]\mathrm{y}_2$ is continuous in both variables together. We shall be concerned with Borel measures in $\mathfrak{M}$ whose null sets are carried into null sets by the homeomorphisms $z \rightarrow [z]y$.

We shall need certain information about the connection between Borel sets in $\mathfrak{M}$ and Borel sets in $\mathfrak{E}$. This information is contained in the two lemmas below.

Lemma (1.2.1)[11]: There exists a Borel set B in $\mathfrak{E}$ such that: (a) B intersects each right G coset in exactly one point and (b) for each compact subset K of $\mathfrak{E}$, $(h^{-1}(h(K))) \cap B$ has a compact closure.

Proof: Choose a compact neighborhood V of the identity e of $\mathfrak{E}$ such that $V = V^{-1}$. If $\mathfrak{E}$ is connected then $\mathfrak{E} = \cup_{n-1}^{\infty} V^n$ and every compact subset of $\mathfrak{E}$ is in some V^n. If $\mathfrak{E}$ is not connected then $\cup_{n-1}^{\infty} V^n$ is an open and closed subgroup with only countably many cosets. In any case it is clear that there exists in $\mathfrak{E}$ a countable family $K_1 \subseteq k_2 \subseteq K_3$; of compact subsets of (M such that every compact subset of $\mathfrak{E}$ is contained in some K_j. By a Theorem of Federer and Morse [14] there exists for each j a Borel subset $B_j \subseteq K_j$ such that $h(B_j) = h(K_j)$ and such that h is one-to-one on B_j. Moreover theB_j may be chosen so that $B_{j+1} \supseteq B_j \; for \; j = 1, 2, ..$ Indeed if $B_1, B_2, \dots, B_j$ have been chosen so that $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_j$ we may define $B_{j+1} \; as \; (B'_{j+1} - h^{-1} \; (h(B_j))) \cup B_j$ where B'_{j+1} is any Borel subset ofk_{j+1} on which h is one-to-one and has range $h(k_{j+1})$.

Since B_j is a Borel set in a complete metric space and h is continuous and one-to-one on B_j It follows from a Theorem in Kuratowski's "Topologie" [26] that $h(B_j)$ is a Borel set and hence that B_{j+l} is a Borel set. Clearly $B = U_{j=1}^{\infty} B_j$has the required properties.

We shall call a Borel subset of $\mathfrak{G}$ with roperties (a) and (b) of the lemma a regular Borel section of $\mathfrak{G}$ with respect to G.

Lemma (1.2.2)[11]: A necessary and sufficient condition that a subset $E \; of \; \mathfrak{M}$ be a Borel set is that $h^{-1}\,(E)$ be a Borel set in $\mathfrak{E}$. A necessary and sufficient condition that a function $f \; on \; \mathfrak{M}$ be a Borel function is that $f \circ h$, where $(f \circ h)\,(x) = f(h(x))$, be a Borel function on $\mathfrak{G}$.

Proof: Let B be a regular Borel section of $\mathfrak{E}$ with respect to $G. If \; h^{-1}\,(E)$ is a B orel set then $h(h^{-1}(E) \cap B) = E$ and is a Borel set by the Kuratowski Theorem referred To above. All other statements of the lemma are obvious or are consequences of this one.

Let μ be a Borel measure in $\mathfrak{M}$; that is a completely additive non negative and plus infinity valued set function defined on all Borel subsets of $\mathfrak{M}$ and finite on compact sets. Suppose that μ is not identically zero.

If for each Borel set $E \subseteq \mathfrak{M}$ and each $y \in, \mathfrak{E}, \mu(E)$ is zero when and only when $\mu([E]y)$ is zero,we shall call μ, a quasi invariant measure. It is easy to see that such measures exist [13].

Indeed let v be any finite, $\not\equiv 0$ Borel measure in $\mathfrak{G}$ whose null sets are those of Haar measure zero and let $\mu\,(E) = v\,(h^{-1}(E))$ for each Borel set $E \subseteq \mathfrak{M}$. Verification of the quasi invariance of $\mu$is immediate. It is our purpose here to study the uniqueness of quasi invariant measures and the analytical properties of the Radon-Nikodym derivatives of their translates. Our results are summarized in Theorem(1.2.6) below. The proof of the Theorem depends upon the three lemmas which follow.

Lemma(1.2.3)[11]: Let μ be any quasi invariant measure in $\mathfrak{M}$. Then $\mu(E) = 0$ if and only if $h^{-1}(E)$ has Haar measure zero.

Proof: Let v denote a right invariant Haar measure in G.For each Borel function f from$\mathfrak{G}$ to the $interval$ [0, 1] $let \; f'(x) = \int_O f(\xi x) \;\; dv(\xi)$. We note that f' is constant on the right G cosets. Let $\mathfrak{J}$ be the family of all functions f under consideration for which f' is also a Borel

function. It is immediate that $\mathfrak{I}$ is closed under the taking f point wise limits and easily seen that it contains all continuous functions with compact support.
Thus $\mathfrak{I}$ contains all Borel functions. Let f'' denote the unique Borel function on $\mathfrak{M}$such that $f'' \circ h = f'$ and for each Borel set $E \subseteq \mathfrak{M}\ let\ \alpha(E) = \int_{\mathfrak{M}} \emptyset''_E(z)d\mu(z)$ where $\emptyset_E$ is the characteristic function of E. Then α is a Borel measure in $\mathfrak{G}$ and$\alpha(E) = 0$ if and only if $\emptyset''_E(z) = 0$ for μ almost all z; that is if and only if $\int \emptyset_E\ (\xi x)dv(\xi) = 0$ for μ almost all h(x); that is if and only if $v([E]_x^{-1} \cap G) = 0$ for μ almost all h(x). On the other hand for each fixed $y \in \mathfrak{H}$we have $a[E]y = 0$ if and only if $v([E]yx^{-1} \cap G) = 0$ for μalmost all $h(x)$. But $v([E]yx^{-l} \cap G) = v([E](xy^{-l})^{-l} \cap G) = g(xy^{-l})$ if $g(x) = v([E]x^{-1} \cap G)$.

Since μ is quasi invariant g(x) is zero for almost all $h(x)$ if and only if this is the case for $g(\mathrm{xy}^{-1})$. Thus $\propto (\mathrm{E}) = 0$ if and only if α([E]y) =0. Thus a is quasi invariant and it follows from [14] Lemma 3 of [27] that α has thesame null sets as Haar measure. Finally it is an easy consequence of the definitions that $\propto (\mathrm{h}^{-1}(\mathrm{E})) = \mu(\mathrm{E})\mathrm{v}(\mathrm{G})$where $0.\infty = 0$. The truth of the lemma follows at once.

Following Weil [42] but interchanging right and left let us write Δ(σ) for the constant Radon Nikody mderivative of the measure E → α (σE) with respect to the measure E → α(E) where α is right invariant Haar measure in $\mathfrak{G}$. Further let us write δ(σ) for the similarly defined constant Radon-Nikodym derivative in G. Δ and a are continuous homeomorphisms of $\mathfrak{G}$and G respectively into the group of positive real numbers.

Lemma (1.2.4)[11]: There exists a positive real valued Borel function p on $\mathfrak{G}$ which is Bounded on compact sets and such that $\rho(\xi x) = (\delta(\xi)/\Delta(\xi))\,\rho(x) forall\ x \in \mathfrak{G}$and all $\xi \in G$

Proof: Let B be a regular Borel section of $\mathfrak{G}$ with respect to G. For each $z \in \mathfrak{M}$ Let $\psi(z)$ be the unique element of B such that $h(\psi/(z)) = z$.

By the Kura-towski Theorem referred to in the proof of Lemma (1.2.1), ψ is a Borel function so that $\psi \circ h$ is also a Borel function. $Let\ \theta_1 = \psi \circ h$. Then $x \longrightarrow x(\theta_1(x))^{-1}$ is a Borel function from $\mathfrak{G}$ to G which we shall denote by θ_2. We now define $\rho(x) = \delta(\theta_2(x))/\Delta(\theta_2(x))$.

We leave it to verify that ρ has the required properties; remarking only that the boundedness of p on compact sets follows from property (b) of regular Borel sections.

We shall call a function with the properties listed in Lemma(1.2.4) a ρ-function. If ρ is any ρ-function on $\mathfrak{G}$then $\rho(xy)/\rho(x)$ is a Borel function of x and y which is constant on the $G \times \mathfrak{G}$ right cosets in $\mathfrak{G} \times \mathfrak{E}$. Since there is a natural homeomorphism of this coset space on $\mathfrak{M} \times \mathfrak{E}$ there is a unique Borel function λ_ρ on $\mathfrak{M} \times \mathfrak{G}$ such that $\lambda_\rho(h(x), y) = \rho(xy)/\rho(x)$ for all x and y in $\mathfrak{G}$. $\lambda_\rho$is easily seen to have the following properties:
(a) for all $z\ in\ \mathfrak{M}$ and all x and $y\ in\ \mathfrak{G}.\ \lambda_\rho(z, xy) = \lambda_\rho([z]x, y)\,\lambda_\rho(z, x)$, (b) for all $\xi\ in\ G, \lambda_\rho(h(e), \xi) = \delta(\xi)/\Delta(\xi)$ where e is the identity of $\mathfrak{G}, (c)\ \lambda_\rho(h(e), y)$ is bounded on compact sets as a function of y. We shall call a positive Borel function on $\mathfrak{M} \times \mathfrak{G}$ with properties (a), (b) and (c) a $\lambda -$ functoin.

It is almost immediate that every λ -function is of the form λ_ρ, for a p-function which is unique except for a positive multiplicative constant. The proof of the next lemma is modeled closely on an argument given by Weil [42] In studying "relatively invariant" measures.

Lemma (1.2.5)[11]: Let ρ be an arbitrary ρ-function on $\mathfrak{G}$. Then there exists a quasi invariant measure $\mu\ in\ \mathfrak{M}$ Such that for all $y\ \in\ \mathfrak{G}\ \lambda_\rho(.,y)$ is a Radon-Nikodym derivative of the measure $E \rightarrow\ \mu([E]y)$ with respect to the measure μ.

Proof: Consider the mapping $f\ \rightarrow\ f"$ defined in the proof of Lemma (1.2.3). As shown by Weil [43] this mapping is "onto" from the continuous functions with compact support in $\mathfrak{G}$ bto the corresponding family of functions in $\mathfrak{M}$.

By virtue of the well-known connection between integrals and measures [25] we may define M by defining $\int\ f"(z)\ d\mu(z)$ for all f with compact support.

We let a denote right invariant Haar measure in $\mathfrak{G}$ and set

$$\int\ f"(z)\ d\mu(z)\ =\int\ f(x)\rho(x)d\alpha(x).$$

In order for this definition to be valid it must be shown that $\int\ f"(z)\ d\mu(z)$ depends only upon $f"$ and not on f . Suppose that $f" \equiv 0$ for some f. We shall show that $\int f(x)\rho(x)da(x)\ =\ 0$. Wehave by definition that $\int f(\xi x)dv(\xi)\ =0$. Hence $\int f(\xi^{-1}x)\delta(\xi^{-1})dv(\xi)\ = O$. Hence for each continuous g With compact support $\iint p(x)g(x)f(\xi^{-1}x)\delta(\xi^{-1})\ dv(\xi)\ da(x)\ =0$. Applying the Fubini Theorem and then $replacing\ x\ by \xi x$ in the integration with respect to x we obtain

$$\iint \rho(\xi x)g(\xi x)f(x)\Delta\,(\xi)\delta(\xi^{-1})da(x)dv(\xi)\ =0\,.$$

Using the ρ -function identity we may eliminate δ and Δ . Following this with a reiointerchange of the order of integration gives

$$\iint \rho(x)g(\xi x)f(x)\ dv(\xi)\ d\alpha(x)\ \ =\ 0\ or\ \int \rho(x)f(x)g'(x)d\alpha(x)=0$$

But as already noted g''can be any continuous function with a compact support in $\mathfrak{M}$. In particular if we choose g so that g'' is one on h(K), where K is the compact support of f, we may conclude that $\int f(x)\rho(x)\ da(x)\ =0$ as desired. Now choose any y in $\mathfrak{G}$ and consider the measure

$E\ \rightarrow\mu([E]y)\ \ =\ \mu_y(E)\int f"(z)\ d_{\mu_y}(z)\ \int f''\big([z]_y^{-1}\big)d\mu(z)=\int f(xy^{-1})\rho(x)d\alpha(x)=$
$\int f(x)\rho(xy)da(x)=\int f(x)\lambda_\rho(h(x),y)\rho(x)d\alpha(x)=\int f"(z)\lambda_\rho(z,y)\ d\mu(z).$

It follows at once that μ is quasi invariant and $\lambda_\rho(.,y)$is aRadon Nikodym derivative of $\mu_y$with respect to μ.

Let us Write $\mu\sim\lambda$ whenever $\lambda(.,y)$ is a Radon derivative othe measure $E\ \rightarrow\mu([E]y)$ with respect to μ for all $y\ \in\ \mathfrak{G}$. Using the above lemmas and the accompanying remarks we should have no difficulty in verifying the truth of the following Theorem.

Theorem (1.2.6)[11]: There exist quasi invariant measures In $\mathfrak{M}$. Any two have the same null sets and hence are mutually absolutely continuous. The Borel set $E\ in\ \mathfrak{M}$ is a null set if and only if $h^{-1}(E)$ has Haar measure zero.

The relations $\mu\sim\lambda$ and $\lambda=\lambda_\rho$ between quasi invariant measures, $\lambda -$ functions and ρ - functions have the following properties:

(a) Every λ -function is of the form $\lambda_\rho;\ \lambda_{\rho1}\ =\ \lambda_{\rho2}$ if and only if$\rho_1/\rho_2$is constant.

(b) For every $\lambda - function\ \lambda$ there is a quasi invariant measure μ such that $\mu \sim \lambda;\ \ if\ \mu_1 \sim \lambda$ and $\mu_2\ \sim\ \lambda$ then μ_1 is a constant multiple of μ_2.
(c) For every quasi invariant measure μ there is $a\ \lambda - function\ \lambda$ such that, $\mu \sim \lambda$; if $\mu \sim \lambda_1$and, $\mu \sim \lambda_2$then for all y, $\lambda_1(.,y) = \lambda_2(.,y)$ almost everywhere in $\mathfrak{M}$.
(d) If $\mu \sim \lambda_{\rho 1}$and, $\mu \sim \lambda_{\rho 2}$ then ρ_1/ρ_2 is almost everywhere constant.
By a representation $U(x\ \rightarrow U_x)$ of the separable4 locally compact group $\mathfrak{G}$ we shall mean a homomorphism of $\mathfrak{G}$ into the group of all unitary trans- formations of some separable [15] Hilbert space $\mathfrak{H}(U)$ onto itself which is continuous in the sense that for each $v\ \in\ \mathfrak{H}(U)$ the function $x \rightarrow\ U_x(v)$ is a continuous function from $\mathfrak{G}\ to\ \mathfrak{H}(U)$. We remind the reader of the well known fact that in order to be able to conclude that U is continuous in this sense it is enough to know that for all v_1 and v_2 in $\mathfrak{H}(U)\ x\ \rightarrow\ (\cup_x(v_1)v_2)$ is a measurable function of x.

Here (,) denotes the scalar product of the expressions inside.

Let G be a closed subgroup of $\mathfrak{G}$and let $L\ (\xi\ \rightarrow\ L\xi)$ be any representation of G. Let μbe any quasi invariant measure in the homogeneous space $\mathfrak{M} =\ \mathfrak{G}/G$ of right G cosets. Let us denote by $\mu_{\mathfrak{H}^L}$ the set of all functions f from $\mathfrak{G}\ to\ \mathfrak{H}(L)$ such that
(a) $(f(x), v)$ is a Borel function of x for all $v\ \in \mathfrak{H}(L)$.
(b) $f(\xi x)\ =\ L_\xi(f(x))$ for all $\xi \in\ G$ and all $x\ \in \mathfrak{G}$.
(c) $\int(f(x), f(x))\ \ d\mu(z)\ \ < \infty$ where the meaning of the integral is to be found in the fact that the integrand is constant on the right G cosets and hence defines a function on $\mathfrak{M} = \mathfrak{G}/G$. It is readily showd that if f_1 and f_2 are any two members of $\mu_{\mathfrak{H}^L}$ then $\int(f_1(x), f_2(x))\, d\mu(z)$ is absolutely convergent. We denote its value by$(f_1{:}\, f_2)$. We shall leave the straightforward but rather tedious task of verifying That when functions equal almost everywhere are identified $\mu_{\mathfrak{H}^L}$ becomes a Hilbert space under the inner product$f_1{:}\, f_2$. It suffices to make obvious modifications in the corresponding proof for the square Summable functions on a measure space and keep in mind Lemma(1.2.2).
Now let ρ be a ρ-function such that $\mu \sim \lambda_\rho$ For each $y\ in\ \mathfrak{G}$ let T_y, map $f\ \in \mu_{\mathfrak{H}^L}$ into g where $g(x)\ =\ \sqrt{\rho(xy)/\rho(x)}f(xy)$.

An obvious calculation shows that g is also in $\mu_{\mathfrak{H}^L}$ and that $(f : f)\ = (g{:}\, g)$. Moreover $T_{v1}(T_{v2}(f)) =\ T_{y_1 y_2}(f)$ Finally an easy argument shows that $(T_y(f){:}\, g)$. is a Borel function of y for each f and g in $\mu_{\mathfrak{H}^L}$. Thus for each y, T_y , defines a unitary transformation $\mu_{U_y^L}$, in the Hilbert in $\mu_{\mathfrak{H}^L}$- space [16] associated with $\mu_{\mathfrak{H}^L}$ and the mapping $y\ \rightarrow \mu_{U_y^L}$ is the representations $\mu_{U^L} of \mathfrak{G}$

Theorem (1.2.7)[11]: Let μ and μ'be quasi is invariant measures in$\mathfrak{M} = \mathfrak{E}/\mathrm{G}$. Then there exists a unitary transformation V from $\mathfrak{H}(\mu_{U^L})$ onto $\mathfrak{H}({\mu'}_{U^L})$ such that $V(\mu_{U_v^L})\, V^{-1}\ = {\mu'}_{U_v^L}$ for all y in $\mathfrak{G}$; that μ' is the representations μ_{U^L} and ${\mu'}_{U^L}$ are unitary equivalent.

Proof: Let ψ with Borel function which is a Radon Nikodym derivative of μ $with$ respect to μ′and let h denote the natural mapping of $\mathfrak{E}$ on $\mathfrak{G}/G$. Then for each $f \in^\mu\ \mathfrak{G}^L, \sqrt{\psi \circ h}(f)$ is in${\mu'}_{\mathfrak{G}^L}$ and the norm of f in $\mu_{\mathfrak{G}^L}$ is equal to that of $\sqrt{\psi \circ hf}$ in $\mu_{\mathfrak{G}^L}$. Moreover every $g$in ${\mu'}_{\mathfrak{G}^L}$ is evidently of the form $\sqrt{\psi \circ hf}$ for some f in $\mu_{\mathfrak{G}^L}$. Let V be multiplication by

$\sqrt{\psi \circ h}$.Then V defines a unitary map of $\mathfrak{G}(\mu_{U^L})$ on ,$\mathfrak{G}(\mu'_{U^L})$. The verification of the fact that $V(\mu_{U_v^L})V^{-1} =^{\mu'} U_v^L$for all y is immediate.

Since we shall not in general distinguish between representations of a group which are unitary equivalent we may drop the μ and refer simply to the representation U^{L} of $\mathfrak{G}$. We shall call U^{L} the representation of $\mathfrak{E}$ induced by the representation L of G. The notation U^L is unambiguous only when no other group containing G as a closed subgroup is under consideration.

Let v denote right invariant Haar measure in G and let C_{L} denote the set of all continuous functions with domain $\mathfrak{G}$, range in $\mathfrak{G}(L)$and compact support. Let K_{f} denote the compact support of the member f of C_{L}.

Lemma (1.2.8)[11]: For each $f \in C_L$.there is a unique function f^0 from $\mathfrak{E}\ to\ \mathfrak{E}(L)$ such that $\int(L_{\xi-I}\,(f(\xi x)),\ v)\,dv(\xi) = (f^0(x), v)\ for\ all\ x \in \mathfrak{G}$ and all $v \in \mathfrak{H}\,(L)$. This function is continuous and is in $\mu_{\mathfrak{G}^L}$ for all quasi invariant measures μ. The function defined on$\mathfrak{G}\backslash G$ by $(f^0(x), f^0(x))$ has a compact support. Finally $sup_{x\in\mathfrak{G}} \| f^0(x) \| \leqq v(K_f K_f^{-1} \cap G)\ sup_{x\in\mathfrak{G}} \| f(x) \|$.

Proof: For each fixed x in $\mathfrak{G}$ consider $\int(L_{\xi-1}(f(\xi x),\ v)\,dv(\xi)$. It is evidently anti- linear and bounded as a function of v and hence is of the form $(f^0(x), v)$ where $f^0(x)$is a member of $\mathfrak{H}(L)$ depending upon x. We must show that the function $f^0(x \to f^0(x))$ has the desired properties. Let K denote the compact support of f.

It follows easily from the definition of f^0 that$\|f^0(x_1) - f^2(x_2)\| \leqq 2(\sup_{\xi\in G}\| f(\xi x_1) - f(\xi x_2)\|)(\sup_{x\in\mathfrak{E}} v(K_x^{-1} \cap G))$ for all x_1 and x_2 in $\mathfrak{G}$. Since f is uniformly continuous it will be sufficient to show the finiteness of $\sup_{x\in\mathfrak{G}} v(\mathrm{Kx}^{-1} \cap \mathrm{G})$ in order to be able to conclude that f^0 is continuous.

Now [17] for all $x \in GK, Kx^{-1} \subseteq KK^{-1}\xi$ for some $\xi \in G$ and hence $Kx^{-1} \cap G \subseteq (KK^{-1} \cap G)\xi$. Moreover for all $x \notin GK, v\,(Kx^{-1} \cap G) = 0$. Thus $v(Kx^{-1} \cap G) \subseteq v(\,KK^{-1} \cap G)$ for all x and hence is bounded as was to be showd. That $f^0(\xi x) = L_\xi f^0(x))$ for all $\xi \in G$ and $x \in \mathfrak{G}$ follows from a straightforward calculation. It is equally easy to see that $(f^0(x), f^0(x)) = 0\ for\ x \notin GK$.Thus as a function $on\ \mathfrak{G}/G, (f^0(x), f^0(x))$ vanishes outside of the compact image of GK in $\mathfrak{G}/G$.

The proof of the final assertion is an obvious modification of that of the continuity of f^0.

We shall denote the class of functions of the form $f^0\ for\ f \in C_L$ by C_L^0.

Lemma (1.2.9)[11]: For each $x \in \mathfrak{G}$ the vectors $f^0(x)\ for\ f^0 \in C_L^0$ form a dense linear sub- space of $\mathfrak{H}(L)$.

Proof: Note first that if $f^0 \in C_L^0$ and f_s is defined by the equation $f_s(x) = f(xs)$ for all x and s in $\mathfrak{G}$ then $(f^0)_s(x) = (f_s)^0(x)$ so that for all f and s, $(f^0)_s \in C_L^0$. Thus the set of vectors $f^0(x)\ for\ f^0 \in C_L^0$ and x fixed is independent of x. Let$\mathfrak{H}_1$ be the orthogonal complement of this set of vectors. Then if $v \in \mathfrak{H}_1$we have $(f^0(x), v) = 0$ for allf^0 and all x. Thus $(f^0\,(\,\xi x), v)) = (f^0(x), L_{\xi-1}(v))$ is zero for all f^0 and x and all $\xi \in G$.

Hence $\mathfrak{H}_1$is invariant under the representation L. Let L' be the component of L in $\mathfrak{H}_1$. Suppose that there exists a non zero member $f^0\ of\ C_L^0$. Then$f^0 \in C_L^0$ and we have a contradiction since the values of $f^0$are all in $\mathfrak{H}_1$. Thus in order to show that $\mathfrak{H}_1 = 0$ and complete the proof of the lemma we need only show that when $\mathfrak{H}_1 \neq 0$ there exists a non zero member f of C_L^0. But if none existed then $\int (L'_{\xi-1}(f(\xi x)), v)\ d\nu(\xi)$ would be zero for all x, all v in $\mathfrak{H}(L)$ and all f in C_L This is readily seen to be impossible.

Lemma (1.2.10)[11]: Let C be any family of functions from $\mathfrak{G}\ to\ \mathfrak{H}\ (L)$ such that:

(a) For some quasi invariant measure $\mu\ in\ \mathfrak{G}/G, C \subseteq \mu_{\mathfrak{H}^L}$

(b) For each $s \in \mathfrak{G}$ there exists a Positive Borel function $\rho_S\ such\ that\ for\ all\ f \in C, P_S f_S \in C$ where $f_S(x) = f(xs)$.

(c) If $f \in C$ then $gf \in C$ for all bounded continuous complex valued functions g on $\mathfrak{G}$ which are constant on the right G cosets.

(d) There exists a sequence $f_1, f_2, \ldots$ of members of C and a subset $P\ of\ \mathfrak{G}$ of positive Haar measure such that for each $x \in P$ the members $f_1(x), f_2(x).,..$ of $\mathfrak{H}(L)$ have $\mathfrak{H}(L)$ as their closed linear span.

Then the members of C have $\mu_{\mathfrak{H}^L}$ as their closed linear span.

Proof: Choose $f_1, f_2, \ldots$ as indicated under (d). Let u be any member of $u_{\mathfrak{H}^L}$ which is orthogonal to all members of C. Then $((\rho_s g)\ (f_j)_s : u)$ is zero for every s and every bounded continuous $g\ on\ \mathfrak{G}$ which is constant on the right G cosets.

If follows at once that for all s and all $j = 1, 2, \ldots\ (f_j(xs), u(x)) = 0$ for almost all $x\ in\ \mathfrak{G}$. Since $(f_j(xs), y(x))$ is a Borel function on $\mathfrak{G}$ we may apply the Fubini Theorem and conclude that for almost all x, $(f_j(xs), u(x))$ is zero for almost all s

Since j runs over a countable class we may select a single null set N in$\mathfrak{G}$ such that for each $x \notin N, \left(f_j(xs), u(x)\right)$ is , for almost all s,zero for all j. It follows that for $x \notin N$ there exists $s \in x^{-1}P$ such that $(f_j(xs), u(x)) = 0\ for\ j = 1,2, \ldots$ and hence that $u(x) = 0$ thus u is almost every where zero and C must be dense in $\mu_{\mathfrak{H}^L}$

Lemma(1.2.11)[11]: Let C_1 be any family of functions from $\mathfrak{H}\ to\ \mathfrak{H}(L)$ such that:

(a) For each $f \in C_1$ there exists a positive Borel function $\rho\ on\ \mathfrak{G}$ such that $(f(x)/\rho(x), v)$ is continuous in x for all $v\ \mathfrak{H}(L)$.

(b) For some quasi invariant measure $\mu\ in\ \mathfrak{G}/G\ , C_1 \leqq \mu_{\mathfrak{H}^L}$

(c) For each $s \in \mathfrak{G}$ there exists a positive Borel function ρ_s, such that for all $f \in C_1, \rho_s\ f_s \in C_1$ where $f_s(x) = f(xs)$.

(d) If $f \in C_1$ then $gf \in C_1$ for all continuous complex valued functions g on $\mathfrak{G}$ which are constant on the right G cosets and vanish outside of $h^{-1}(K)$ for some com- pact subset K of $\mathfrak{G}/G$.

(e) For some (and hence all) $x \in \mathfrak{G}$ the members $f(x)\ of\ \mathfrak{H}\ (L)$ for $f \in C_1$ and x fixed have $\mathfrak{H}\ (L)$ as their closed linear span.

Then the members of C_1 have $\mu_{\mathfrak{H}^L}$ as their closed linear span.

Proof: Choose $f_1, f_2, \ldots\ in\ C_1$ so that $f_1(e), f_2(e), \ldots$ have $\mathfrak{H}\ (L)$ as their closed linear span; e being the identity of $\mathfrak{E}$. Let u be any member of $\mu_{\mathfrak{H}^L}$ which is orthogonal to all members of C_1. Then $(\rho_s\ g(f_j)_s : u)$ is zero for every $s \in \mathfrak{G}$ and every g which satisfies the conditions listed under (d). It follows at once that for all x and all $j = 1, 2,\ , (f_j(xs), u(x)) = 0$ for almost all x in$\mathfrak{G}$.

Since $(f_j(xs), u(x))$ is a Borel function on $\mathfrak{G} X \mathfrak{G}$ we may apply the Fubini Theorem and conclude that for almost all $x, (f_j(xs), u(x))$ is zero for almost all s. Since j runs over a countable class we may select a single null set $N\ in\ \mathfrak{G}$ such that for each $x \notin N$, $(f_j(xs), u(x))$ is for almost all s zero for all j. Suppose that $u(x_1) \neq 0$ for some $x_1 \notin N$ then $(f_j(e), u(x_1)) \neq 0$ for some j. But for some positive$\rho, (f_j(x), u(x_1))/\rho(x)$ is continuous in x.Hence $(f_j, (x_1 s), u(x_1))/\rho(x_1 s) \neq 0$ for s in some neighborhood of x_1^{-1}. Hence $(f_j(xs), u(x_1)) \neq 0$ for s in some neighborhood of x_1^{-1}.But this contradicts the fact that $(f_j(xs), u(x_1))$ is almost everywhere zero.

Thus $u(x)$ is zero almost everywhere. Thus only the zero element is orthogonal to all members of C_1 and it follows that C_1 must be dense.

Lemma (1.2.12)[11]:C_1^0 is dense in $\mu_{\mathfrak{H}^L}$ for all quasi invariant measures$\mu\ on\ \mathfrak{G}/G$.

Proof: The truth of the lemma is an immediate consequence of Lemmas (1.2.8), (1.2.9) and (1.2.11).

Let there be given two closed subgroups G_1 and G_2 of $\mathfrak{G}$ such that $G_1 \subseteq G_2$.

Let L be a representation of G_1 . Then we may form ${G_2}^{U\ L}$ and $\mathfrak{G}^{U\ L}$. Denoting the first of these representations by M we may also form $\mathfrak{G}^{U\ M}$.

Our object is to show [18] that the representations U^{M} and U^{L} are unit ary equivalent; in words that one may "raise L from G_1 up to $\mathfrak{G}$" in several stages without affecting the result. To do this let μ_1 , and μ_2 be arbitrary quasi invariant measures in $\mathfrak{G}/G_1$ and $\mathfrak{G}/G_2$ Respectively and let ρ_1 and ρ_2 be associated ρ -functions. Then let us define $\rho_3(x)\ as\ \rho_1(x)/\rho_2(x)$ for all $x\ in\ \mathfrak{G}$ and let δ_1 and δ_2 be defined for G_1 and G_2 as δ was for G.

We see at once the $\rho_3(\xi x) = \rho_1(\xi x)/\rho_2(\xi x) = \frac{\delta_1(\xi)}{\Delta(\xi)}\ \rho_1(x)/\frac{\delta_2(\xi)}{\Delta(\xi)}\ \rho_2(x) = \frac{\delta_1(\xi)}{\delta_2(\xi)}\ \rho_3(x)$ for $\xi \in G_1$ and $x \in \mathfrak{G}$. Thus ρ_3 restricted to G_2 is a ρ-function for the homogeneous space G_2/G_1. We let μ_3 be a quasi invariant measure associated with this ρ-function. In what follows v_1, , v_2 and v will denote right invariant Haar measure in G_1, G_2 and $\mathfrak{G}$ respectively. When f is a function on G_2 which is con-stant on the right G_1, cosets we shall use the notation $\int f(y)\, d_{\mu_3}(z)$ to indicate the integral with respect to$\mu_3$of the function defined $G_2/G_1$by f and likewise for $\mu_1$and μ_2.

We shall also use $\mathfrak{G}^{C_L^0}$ etc. to distinguish between the possible meanings of C_L^0 in the present context. Now for each fixed $x\ in\ \mathfrak{G}$ and each f in for ${\mu_1}_{\mathfrak{H}^L}$ let f_x denote the function from $G_2\ to\ \mathfrak{H}(L)$ which takes $y \in G_2$ into $f(yx)\sqrt{\rho_3(yx)/\rho_3(y)}$. We show first that for f in a certain dense subspace of ${\mu_1}_{\mathfrak{H}^L}$we have $f_x \in {\mu_1}_{\mathfrak{H}^L}$ for all x . It is evident that for any $f \in {\mu_1}_{\mathfrak{H}^L}$we have $f_x(\xi y) = L_\xi\big(f_x(y)\big)$ for all $\xi \in G_1$ and all $y \in G_2$. Moreover

$$(f_X(y), w) = (f(yx), w)\sqrt{\rho_3(yx) / \rho_3(y)}$$

which is surely a Borel function of y for all $w \in \mathfrak{H}(L)$. Finally we show that if$f$ is of the form $U_{s_1}^L(F_1) + U_{s_2}^L(F_2) + \ldots + U_{s_n}^L(F_n)$ where each $F_j \in C_L^0$ and each $S_j \in \mathfrak{G}$ then for all $x, \int (f_x(y), f_x(y))\, d\mu_3(z) < \infty$ so that f_x is in , ${\mu_3}_{\mathfrak{H}^L}$ .To do this we need only consider the case in which f is actually of the form $U_s^L(F)$ for an $F \in C_L^0$.

Since we shall have use a little later for the resulting formula we shall compute $\int (f_x(y), f_x'(y))\, d\mu_3(z)$ where $f' = U_{s'}^L(F')$ and s' and F' may be different from s and F. It is not clear that this integral has meaning unless the integrand is non negative. However its finiteness in the case $s = s'$ and $F = F'$ implies its meaningfulness and finiteness in the general case. Thus we may compute it on the assumption that the integrand is summable and this assumption will be justified by the fact that the endresult will be evidently finite in all cases.

We have

$$\int (fx(y), f'(y))\ d\mu_3(z) = \int (\rho_3(yx)/\rho_3(y))(f(yx),$$

$$f'(yx))\, d\mu_3(z) = \int \rho_3(yx)/\rho_3(y))\,(\sqrt{\rho_1(yxs)\rho_1(yxs')}/\rho_1(x))\,(F(yxs),$$

$F'(yxs')d\mu_3(z)$ using the fact that $\rho_1(yxs)/\rho_1(x) = \rho_2(yxs)\rho_3(yxs)\ /\ \rho_2(yx)\rho_3(yx)$ and $\rho_2(yxs)/\rho_2(yx) = \rho_2(xs)/\rho_2(x)$ this reduces to

$$\sqrt{\rho_2(xS)\rho_2(xs')}\ /\ \rho_2(x))\ \int \sqrt{\rho_3(yxs)\rho_3(yxs')}\,/\rho_3(y))(F(yxs), F'(yxs'))\ d\mu_3(z).$$

Now by Lemma(1.2.8) $(\mathrm{F(xs)}, \mathrm{F'(xs')}$ as a function of x defines a continuous function with compact support on $\mathfrak{G}/G_1$. Thus by the argument of Weil referred to in the proof of Lemma (1.2.5) it may be put in the form $\left(\mathrm{F(xs)}, \mathrm{F'(xs')}\right) = \int \psi(\xi x)\mathrm{d}v_1\xi$ where ψ is continuous on $\mathfrak{G}$ and has compact support. Thus our expression may be written [19]

$$(\sqrt{\rho_2(xs)\rho_2(xs')}/\rho_2(x))\ \int \sqrt{\rho_3(yxs)\rho_3(yxs')}\ /\rho_3(y))\ \int \psi\ (\xi yx)\ dv_1(\xi)\ d\mu_3(z)$$

$$= (\sqrt{\rho_2(xs)\rho_2(xs')}/\rho_2(x))\ \int \sqrt{\rho_3(yxs)\rho_3(yxs')}\ \psi\ (yx)dv_2(y)$$

and this is evidently finite.

Let us designate the set of all f' s of the form $U_{s_1}^L(F_1) + \ldots + U_{s_n}^L(F_2)$ by C_L^T. Our next task is to show that for each $f \in C_L^T$ the function $x \to f_x$ from $\mathfrak{G}$ to $\mu_{3_{\mathfrak{H}^L}}$ is in $\mu_{2_{\mathfrak{H}^M}}$ and has the same norm as amember of $\mu_{2_{\mathfrak{H}^M}}$ that f foes when regarded as a member of $\mu_{2_{\mathfrak{H}^L}}$. First of all if f is any element of C_L^T and $g \in \mu_{3_{\mathfrak{H}^L}}$ then $(f_x : g) = \int \sqrt{\rho_3(yx)\rho_3(y)}\ \left(f\ (yx), g(y)\right)d\mu_3(z)$ and this is clearly a Borel function of x. Moreover

$$f_\eta x(y) = f(y\eta x)\sqrt{\rho_3(y\eta x)\ /\rho_3(y)}$$

$$= f((y\eta)x)\sqrt{\rho_3(y\eta x)/\rho_3(y\eta)}\sqrt{\rho_3(y\eta)/\rho_3(y)}\ f_x(y\eta)\sqrt{\rho_3(y\eta)/\rho_3(y\,)}.$$

For all x in $\mathfrak{G}$ and all $\eta$and y in G_2. Thus $f_{\eta x} = M_n(f_x)$Finally in order to establish the fact that $\int (f_x : f_x)\ d\mu_2(z)$ is finite and equal to $\int (f(x), f(x)\ d\mu_1(z)$we need only show that if $f \in_{\mathfrak{E}} U_s^L(F)$ and $f' = f \in_{\mathfrak{E}} U_{s'}^L(F')$ where F and F' are in $\mathfrak{G}^{C_L^0}$ then $\int (f_x : f_x')\ d\mu_2(z) = \int (f(x), f'(x)\ \ d\mu_1(z)$. Moreover just as in the corresponding situation above we may assume that the first Integrand is summable. Using the expression computed above for $f_x : f_x'$we get

$$\int (f_x : f_x')\ d\mu_2(z)$$

$$= \int (\sqrt{\rho_2(xs)\rho_2(xs')}/\rho_2(x))\ \int \sqrt{\rho_3(yxs)\rho_3(yxs')}\ \ \psi(yx)dv_2(y)d\mu_2(z)$$

$$= \int \sqrt{\rho_2(xs)\rho_2(xs')}\ \sqrt{\rho_3(xs)\rho_3(xs')}\,\psi(x)\ dv(x)\ =\ \sqrt{\rho_1(xs)\rho_1(xs')}\psi(x)\ dv(x).$$

On the other hand

$$\int (f(x), f'(x))\ d\mu_1(z)\ =\ \int \sqrt{(\rho_1(xs)\rho_1((xs')}\ /\rho_1(x))\ F(xs), F'(xs')d\mu_1(z)$$

$$= \int \sqrt{(\rho_1(xs)\rho_1((xs')}\ /\rho_1(x))\ (\int \psi\,(\xi x)\ dv_1(\xi))d\mu_1(z)$$

$$= \int \sqrt{(\rho_1(xs)\rho_1((xs')}\,\psi(x)dv(x).$$

Thus the desired equality is established.

We have now a linear norm preserving map from the dense subspace C_{L}^{T} of $\mu_{1_{\mathfrak{H}^L}}$ *into* $\mu_{2_{\mathfrak{H}^L}}$. We denote by T its unique continuous extension to all of T then is a unitary map of $\mu_{1_{\mathfrak{H}^L}}$ onto a closed subspace of $\mu_{2_{\mathfrak{H}^M}}$. In order to Complete the proof of the unitary equivalence of U^{L} and U^{M} we must show that $T_{\mathfrak{G}}U_x^L = \mathfrak{G}^{U_X^M}$ for all x in $\mathfrak{E}$ and that T has all of $\mu_{2_{\mathfrak{H}^M}}$ for its range.

An obvious computation shows that $T_{\mathfrak{G}}U_x^L = \mathfrak{G}^{U_X^M}\ T(f\ )$ for all f in C_{L}^{T} and the corresponding result for general f follows from the density of C_{L}^{T}. To show that the range of T is all of $\mu_{2_{\mathfrak{H}^M}}$ we have only to show that this range is dense in $\mu_{2_{\mathfrak{H}^M}}$ and this may be done using Lemmas(1.2.10) and (1.2.11). Let C of Lemma (1.2.10) be the set of all $T(f)$ for $f \in C_L^T$. Then (a) of this lemma is clearly satisfied. That (b) and (c) are also satisfied follows from quite elementary considerations. It remains to verify (d). To this end we first apply Lemma (1.2.11) to show that for each $x\ \in\ \mathfrak{G}$ the set of all members of $\mu_{3_{\mathfrak{H}^M}}$ of the form f_x for $f\ \in\ C_L^0$ is dense in $\mu_{3_{\mathfrak{H}^M}}$ given $x\ \in\ \mathfrak{G}$, let C_1 of this lemma be the set of all members of $\mu_{3_{\mathfrak{H}^M}}$ of the form f_x for $f \in\ C_L^0$. If $f_x\ \in\ C_1$ then $f_x(y) = f(yx)\sqrt{\rho_3(xy)\ /\ \rho_3(y)}$. Since by Lemma (1.2.8), $(f(yx), w)$ is continuous in y for each w in $\mathfrak{H}(L)$ it follows that (a) of Lemma(1.2.11) is satisfied. Condition (b) has already been verified and conditions (c) and (d) may be verified by obvious arguments. That condition (e) is satisfied follows immediately from Lemma (1.2.9). Thus Lemma(1.2.11) applies and for each $x\ \in \mathfrak{G}$ the f_x for $f\ \in_{\mathfrak{G}}\ C_L^0$ are indeed dense in $\mu_{3_{\mathfrak{H}^L}}$. Now choose a sequence $f_1\ , f_2$, ... of members of $\mathfrak{G}^{C_L}$ such that for each $f\ \in_{\mathfrak{G}}\ C_L^0$ there exists a subsequence $f_{n_1}\ , f_{n_2}$, ...which converges uniformly to f and is such that its members vanish outside of a common compact set. We shall show that the sequence $T(f_1^0), T(f_2^0)$, of members of $\mu_{2_{\mathfrak{H}^M}}$ has the property required in (d) of lemma (1.2.10); that we shall show that for each $x\ in \mathfrak{G}$ the members $y \rightarrow f_j^0(xy)\sqrt{\rho_3(yx)\ /\ \rho_3(y)}$ of $\mu_{3_{\mathfrak{H}^L}}$ are dense in $\mu_{3_{\mathfrak{H}^L}}$. In view of the foregoing we need only show that $if\ x\ \in\ \mathfrak{G}$ and $f_{nk}\ \rightarrow\ f$ in the sense indicated above then $\left(f_{n_k}^0\right)_x \rightarrow\ (f^0)_x$ in the $\mu_{3_{\mathfrak{H}^L}}$ norm.

Thus we need only show that $\|f_x^0\| \leq M(x.K)sup_{s\in\mathfrak{G}}\|f(s)\|$ where K is a compact set containing the support of f and $M(x.K)$ is a positive real number. But $\|f_x^0\| = \int\big(f^0(yx), f^0(yx)\big)\,(\rho_3(yx)/\rho_3(y))\ d\mu_3(z)$ and if f vanishes outside K and $h_2\ maps\ G_2$ canonically on G_2/G_1 then $f(yx)$ vanishes for $y\ \in\ K_x^{-1} \cap G_2\ s0f^0(yx)$ vanishes for

$h(y) \notin h(K_x^{-1} \cap G_2)$. Let ψ be continuous and have compact support in G_2 and be such that $\psi_1(y) = \int \psi(\xi y) dv(\xi)$ is one on $h(K_x^{-1} \cap G_2)$. Then

$$\|f_x^0\|^2 \leqq \left(sup_{u\in g}\|f^0(u)\|^2\right) \int \psi(y)\rho_3(yx)dv_2(y)$$

$$\leq v_1(KK^{-1} \cap G_1) \int \psi(y)\rho_3(yx)dv_2(y)\left(sup_{u\in g}\|f(u)\|^2\right).$$

We have thus completed the proof.

Theorem (1.2.13)[11]: If L is the regular representation of $G_{2\subseteq}$ $\mathfrak{G}$ then , $\mathfrak{G}U^L$ is the regular representation of $\mathfrak{G}$.

Let us denote the conjugate space of a Hilbert space $\mathfrak{H}$ by $\overline{\mathfrak{H}}$. We know of course that there is a standard norm preserving anti linear mapping $v \rightarrow v^*$ of $\mathfrak{H}$ onto $\overline{\mathfrak{H}}$; This being defined by the equation $v^*(w) = (w, v)$. Nevertheless it will be convenient to distinguish between $\mathfrak{H}$ and $\overline{\mathfrak{H}}$ (although not between $(\mathfrak{H}$ and $\overline{\mathfrak{H}})$).

We have thus two meanings to be attached to the adjoint T^* of a bounded linear operator $T\ in\ \mathfrak{H}$; the ordinary general meaning as an operator in $\overline{\mathfrak{H}}$ and the specific Hilbert space meaning $T^*(v) = (T^*(v^*))^*$. We shall use T^* in both cases trusting to the context or explanatory remarks to make clear what is meant in each instance.

Let Ube an arbitrary representation of the separable locally compact group $\mathfrak{H}$. By $\overline{U}$ the adjoint of U we shall mean the representation of $\mathfrak{G}\ in\ \overline{\mathfrak{H}(U)}$ defined by the equation $(\overline{U})_x = (U_{x-1})^*$

Now let L be a representation of the closed subgroup $G\ of\ \mathfrak{H}$. If ,μ is a quasi invariant measure in $\mathfrak{G}/G$ and f and g are members of $\mathfrak{H}(\mu_{U^L})$ and $\mathfrak{H}\left((\mu_{U^{\overline{L}}})\right)$ respectively then for each $x \in \mathfrak{G}$ $(f(x), g(x)^*)$ is a well defined complex number.

We have in addition $(f(\xi x), g(\xi x)^*) = (f(x), g(x)^*)$ for all $\xi\ in\ G$. Thus since $(f(x), g(x)^*) \leqq \|f(x)\|\ \ \|g(x)\|$ and $\|f(x)\|^2$ and $\|g(x)\|^2$ defines define summable functions on $\mathfrak{G}/G$ we may form$\int (f(x), g(x)^* d\mu(z) = f|g$ Thus each $g \in \mathfrak{H}(\mu_{U^L})$. Defines a member $f \rightarrow f|g of\ \overline{\mathfrak{H}(\mu_{U^L})}$.

Theorem (1.2.14)[11]: The meaning of $\mathfrak{H}(\mu_{U^{\overline{L}}})$ into $\overline{\mathfrak{H}(\mu_{U^L})}$defined by the equation $f|g = \int (f(x), g(x)^* d\mu(z)$ is onto and unitary. It sets up a unitary equivalence between the representations $U^{\overline{L}}$ and $\overline{U^L}$.

Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$be two Hilbert spaces and let T be a linear transformation from $\overline{\mathfrak{H}}_2\ to\ \mathfrak{H}_1$. Then T^* will be a linear transformation from$\overline{\mathfrak{H}}_1\ to\ \mathfrak{H}_2$.

Let A be defined by the equation $A(v) = (T * (T(v)^*))^*\ for\ all\ v \in \overline{\mathfrak{H}}_2$. . Then A is a non negative self adjoint operator in $\overline{\mathfrak{H}}_2$ and admits accordingly a (possibly infinite) trace. We let $|||T||| = \sqrt{TraceA}$ .As is well known and easily verified the set of all T for which $|||T||| < \infty$ is a Hilbert space under the norm $\rightarrow\ |||T|||$.

The corresponding inner product is given by (T,S) = Trace B were $B(v) = (S^*(T(v)^*))^*$. The T's in this Hilbert space are called the Hilbert-Schmidt operators from $\overline{\mathfrak{H}}_2$and $\mathfrak{H}_1$. and the Hilbert space itself the Kronecker product $\mathfrak{H}_1 \times \mathfrak{H}_2 of\ \mathfrak{H}_1$and $\mathfrak{H}_2$

Now let U and V be representations of the separable locally compact groups $\mathfrak{H}_1$and $\mathfrak{H}_2$.

For each $x_1, x_2 \in \mathfrak{G}_1\ X\ \mathfrak{G}_2$ the mapping $T \rightarrow\ U_{x_1}, T(V_{x_2})^*$ is a unitary transformation of $\mathfrak{H}(U)\ \times\ \mathfrak{H}(V)$onto itself. We shall denote it by $(U\ X\ V)_{x_1,x_2}$.

Clearly the map $x_1, x_2 \rightarrow (U \times V)_{x_1,x_2}$ is a unitary representation of $\mathfrak{G}_1 \times \mathfrak{G}_2$.We shall call it the outer Kronecker product $U \times V$ of the representations U and V of $\mathfrak{G}_1 X \mathfrak{G}_2$.if $\mathfrak{G}_1 = \mathfrak{G}_2 = \mathfrak{G}$ then the subgroup $\widetilde{\mathfrak{G}}^2$ of all $x, y \, in \, \mathfrak{G}_1 \times \mathfrak{G}_2 = \mathfrak{G} X \mathfrak{G} \, with \, x = y$ is isomorphic to $\mathfrak{G}$. The restriction to $\widetilde{\mathfrak{G}}^2$ of $U X V$ thus defines a representation Of $\mathfrak{G}$ which we shall call the Kronecker product $U \otimes V$ of the representations U and V of $\mathfrak{G}$. We note that $T \rightarrow T^*$ sets up a unitary equivalence between $U \otimes V$ and $V \otimes U$.

Theorem (1.2.15)[11]: Let L and M be representations of the closed subgroups G_1 and G_2 of the separable locally compact groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$ respectively.

Then the representations $\mathfrak{G}_1 \times \mathfrak{G}_2{}^{U^{L\times M}} \, and \, {}_{\mathfrak{G}_1} U^L \times {}_{\mathfrak{G}_2} U^M \, of \, \mathfrak{G}_1 \times \mathfrak{G}_2$ are unitary equivalent.

Proof: Let T be a member of $\mathfrak{H}(U^L \times U^M)$ [that is an operator from $\overline{\mathfrak{H}(\mu^2 \mathrm{U}^M)}$ to $\mathfrak{H}(\mu_1 U^L)$] whose range is finite dimensional.Then there exist $f_1, f_2, \dots f_n \in H({}^{\mu_1}U^L)$ and $g_1, g_2 \dots . g_n \in \, , \mathfrak{H}({}^{\mu_2}U^M)$ such that for each $g \in \, , \mathfrak{H}({}^{\mu_2}U^M)$ we have $T(g^*) = (g_1, g)f_1 + \cdots + \underline{(g_n, g)}f_n$. For each $x, y \in \mathfrak{G}_1 \times \mathfrak{G}_2$ we may define an operator $A_T(x, y)$ from$\overline{\mathfrak{H}(M)} \, to \, \mathfrak{H}(L)$ as follows. $(A_T(x, y)(w^*) = f_1(x)(g_1(y), w) + \cdots + f_n(x)(g_n(y), w)$.
We note at once that $A_T(\xi x, \eta y) = L_\xi A_T(x, y)M_\eta^* \, for \, all \, x, y \in \mathfrak{G}_1 \times \mathfrak{G}_2$ and all$\xi, n \in G_1 \times G_2$.

Moreover $|||A_T(x, y)|||^2 = \sum_{ij}\left(f_i(x), f_j(x)\right)(g_i(y), g_i(y))$ and

$$|||T|||^2 = \sum_{ij}(f_i : f_j)(g_j : g_i) = \sum_{ij}\left(\int (f_i(x), f_j(x))d\mu_1(z)\right)\left(\int (g_j(y), g_i(y)d\mu_2(z)\right)$$

$$= \int \left(\sum_{i,j}(f_i(x), f_j(x)\right)(g_j(y), g_i(y))d(\mu_1 \times \mu_2)(z)$$

$$= \int |||A_T(x, y)|||^2 \, d(\mu_1 \times \mu_2)(z).$$

Thus the function $A_T(x, y \rightarrow A_T(x, y))$ is a member of $\mathfrak{H}(U^{LXM})$ and the mapping $T \rightarrow A_T$ is linear and norm preserving. Moreover the domain and range of this mapping are dense in $\mathfrak{H}(U^L \, X \, U^M)$ and$\mathfrak{H} \, (U^{LXM})$ respectively.
As far as the domain is concerned this follows from the theory of Hilbert-Schmidt operators. To show that the range is dense we need only apply Lemma(1.2.11) to those particular $A_T{}'S$ for which the f_i and g_j are in $\mathfrak{G}_1 C_L^0$ and,$\mathfrak{G}_2 C_M^0$ respectively.We leave the easy but mildly tedious task of verifying that the hypotheses of this lemma are satisfied. $T \rightarrow A_T$ may thus be extended by continuity to give a unitary map of $\mathfrak{H}(U^L \, XU^M) \, On \, \mathfrak{H}(U^{LXM})$ and it is almost immediate that this map sets up the required unitary equivalence.

Corollary (1.2.16)[11]: Let the separable locally compact group $\mathfrak{G}$ be the direct product $G \times G_1$ of the closed subgroups G and G_1 and let L be a representation of G.
Then U^{L} is unitary equivalent to the outer Kronecker product of L with the regular representation of G_1.

The second of our three main Theorems asserts the existence of a certain useful direct sum decomposition of the Kronecker product $U^L \otimes U^M$ of two induced representations of a group $\mathfrak{G}$. By definition$U^L \otimes U^M$ is obtained from the outer Kronecker product $U^L X \, U^M \, of \, \mathfrak{G} X \, \mathfrak{G}$ by restricting $U^L \, XU^M$ to the subigroup $\mathfrak{G} \, of \, all \, x, y \in \mathfrak{G} \, X \, \mathfrak{G} \, with \, x = y$. By Theorem

(1.2.15) , U^{LXM} is unitary equivailent to $U^L XU^M$. Thus $U^L XU^M$ can be analyzed by ahalyzing the restriction of $U^{\ LXM} to\ \bar{\mathfrak{G}}$.Our Theorem on Kronecker products follows from these remarks and a Theorem (our first main Theorem) on the decomposition of the restriction of an induced representation to a closed subgroup. Let L be a representation of the closed subgroup $G_1\ of \mathfrak{G}$ and consider the restriction $U^{L,G_2}\ of\ U^L$ to a second closed subgroup G_2.

While $\mathfrak{G}$ acts transitively on the homogeneous space $\mathfrak{G}/G_1$ or right G_1 cosets this will not be true in general of G_2.

Moreover any division of $\mathfrak{G}/G_1$ into two parts S_1 and S_2,each a Borel set which is not a null set[9], and each invariant under G_2 leads to a corresponding direct sum decomposition of U^{L,G_2} . indeed the closed subspaces $\mathfrak{H}_{s_1}$and $\mathfrak{H}_{s_2}$ of all $f\ \in\ \mathfrak{G}(\mu_{U^L})$ which vanish outside of $h^{-1}(S_1)$ and $h^{-1}(S_2)$ respectively are invariant and are orthogonal Complements of one another.

Suppose that there is a null set N in $\mathfrak{G}/G_1$ whose complement is the union of countably many non null orbits $C_1, C_2, \ldots\ of\ \mathfrak{G}/G_1$. under G_2. Then we obtain by the above procedure a direct sum decomposition of U^{L,G_2} into as many parts as there are non null orbits. Our Theorem follows from an analysis of the nature of these parts and it is with this analysis that the present is concerned.

We consider a more general case in which all of the orbits can be null sets and our sum becomes an integral.

Since we can do so with but little extra effort we will make the analysis to follow apply to this case as well. Of course according to the definition given above $\mathfrak{H}_c$ will be zero dimensional whenever C is a null orbit. However it is possible to reword the definition so that it always gives a non zero Hilbert space and so that when C is not a null set this definition is essentially the same as that already given.

Indeed note that when C is an orbit which is not a null set then $\mathfrak{H}_c$may be equivalently defined as follows.

Let x_0 be any member of $\mathfrak{G}$ such that $h(x_0)\ \in\ C$ and consider the set $\mathfrak{H}_c^1$ of all functions f from the double coset $G_1x_0G_2\ to\ \mathfrak{H}(L)$ such that:(a) (f(x),v) is a Borel function for all $v\ \in \mathfrak{H}(L),\ = (b)\ f(\xi x) = L_\xi\big(f(x)\big)$ for all $\xi\ \in\ G_1$ and all

$x\ \in\ G_1x_0G_2$ and $(c)\ \int_c\ (f(x), f(x))d\mu(z) < \infty$ where μ is a quasi-invariant measure in $\mathfrak{G}/G_1$.

$\mathfrak{H}_c^1$ under the norm implicitly defined under (c) is evidently isomorphic to $\mathfrak{H}_c$ in a natural manner. Moreover the measure in C need not be defined by restricting $\mu\ to\ C$. Instead noting that G_2 acts transitively on C we may apply Theorem(1.2.6) and define $\mu_c$as a quasi invariant measure in C associated with the $\lambda\ function\ \lambda_c, where\ \lambda_c,$ is the restriction to $C\ X\ \mathfrak{G}_2$ of theλ-function associated with μ. Strictly speaking Theorem (1.2.6) does not apply since C is not a coset space or even known to be locally compact.

However the mapping $x\ \rightarrow\ h(x_0x)$ sets up a one-to-one Borel set preserving Correspondence between the points of C and those of the coset space G_2/G_0 Where G_0 is the set of all $x\ \in\ G_2$such that $h(x_0x) =\ h(x_0)$; that is $G_0\ =\ G_2 \cap (x_0^{-1}G_1x_0)$.Moreover it is evident that this mapping allows us to apply Theorem(1.2.6) to the case at hand. Using

$\mu_c\ for\ \mu$ the above - definition of $\mathfrak{H}_c'$ gives a non trivial space for every orbit C. We are now in a position to formulate the lemma giving the sought for analysis.

Lemma(1.2.17)[11]: Let C be any orbit in $\mathfrak{G}/G_1$ under G_2 and let x_o be such that $h(x_0) \in C$.

Let $\mathfrak{H}_c'$be defined as above .Let U be the representation of G_2 induced by the representation $\eta \to L_{x_0\eta x_0^{-1}}$of $G_2 \cap (xx_0^{-1}G_1x_0)$. Then there is a unitary map of $\mathfrak{H}$(U) on $\mathfrak{H}_c^1$ such that if $f \in \mathfrak{H}(U)$ corresponds to $g \in \mathfrak{H}_c^{1}{}'$ then $U_s(f)$ corresponds tog_s where $g_s(x) = g(xs)\sqrt{\lambda(x,s)}$ for all $x \in C$ and all $s \in G_2$.

Proof: For each function f on $G_1x_0G_2$ which satisfies conditions (a) and (b) of the definition of $\mathfrak{H}_c'$ let $\tilde{f}(t) = f(x_0t)\ for\ all\ t \in G_2$.

Then $(f(t),v)$ is a Borel function on G_2 for all $v \in \mathfrak{H}(L)$. Moreover if $\eta \in G_0 = G_2 \cap (x_0^{-1}\,G_1x_0)$ then if $\xi = x_0\eta x_0^{-1}$ we have $\tilde{f}(\eta t) = \tilde{f}(x_0^{-1}\xi x_0 t) = f(\xi x_0 t) = L_\xi\,\tilde{f}\,(t) = L_{x_0\eta x_0^{-1}}\,(\tilde{f}(t))$ that is

$$(*)\tilde{f}(\eta t) = L_{x_0\eta x_0^{-1}}\,(\tilde{f}(t))$$

for all $t \in \mathfrak{G}_2$ and all $\eta \in G_2 \cap (x_0^{-1}G_1x_0)$. Conversely let g be any function from $G_2 to\ \mathfrak{H}(L)$ which is a Borel function in the sense that $\tilde{f}$ was and which satisfies (*).We define $f(\xi x_0 t) = L_\xi(\,(g\ t))$ for all $\xi \in G_1$ and $t \in G_2$. If $\xi_1x_0t_1 = \xi_2x_0t_2$ then $\xi_2^{-1}\ \xi_1 = x_0t_2t_1^{-1}x_0^{-1}$ so that $g(t_2t_1^{-1}t) = L_{\xi_2\xi_1}^{-1}(g(t))$. Thus $L_{\xi 2}\big(g(t_2)\big) = L_{\xi 1}\big(g(t_1)\big)$ and f is unambiguously defined. We show next that $(f(x),v)$ is a Borel function of x for all $v \in \mathfrak{H}(L)$. Let $f_1(\xi,\eta) = L_\xi(g(\eta))$ for all ξ,η in G_1, G_2 .

Then $(f_1(\,\xi,\eta),v) = (g(\eta),L_{\xi-1}(v)) = \sum_{i=1}^{\infty}(g(\eta),\varphi_i)\,\big(\varphi_i,L_{\xi-1}(v)\big)$ where $\{\varphi_i\}$ is an orthonormal basis for $\mathfrak{H}(L)$.

Evidently then $(f_1(\,\xi,\eta),v)$ is a Borel function of $\xi,\eta \in G_1X\,G_2$.

Let us introduce a new group operation in $G_1X\,G_2$. by defining $(\xi_1,\eta_1)(\xi_2,\eta_2) = (\xi_1\xi_2,\eta_2\eta_1)$and let us call the resulting group G_3.

Then $\xi_1x_0\eta_1 = \xi_2x_0\eta_2$ if and only if $(\xi_2,\eta_2)^{-1}(\xi_1,\eta_1) = \xi_2^{-1}\xi_1,\eta_1\eta_2^{-1}$ has the form $\xi, x_0^{-1}\xi^{-1}x_0$.

The set of all such is a subgroup $G_4\ of\ G_3$ .Thus $\xi,\eta \to \xi x_0\eta$ sets up a one-to-one correspondence between the points of the homogeneous space G_3/G_4 and the points of the double coset $G_1x_0G_2$.

Moreover it follows from the existence of Borel sections that the function on G_3/G_4 defined by $(f_1(\xi,\eta),v)$ is a Borel function.

That $(f(x),v)$ is a Borel function now follows from the fact that the mapping of $G_3/G_4$0f on $G_1x_0G_2$ defined above preserves Borel sets.

Observe finally that $\tilde{f} = g$ is a one-to-one map of Borel functions satisfying (*) onto functions satisfying (a) and (b) of the definition of $\mathfrak{H}_c'$ .Consider the mapping $t \to h\,(x_0t)$ of $G_2\ onto\ C$. It is one-to-one and Borel set preserving from $G_2/(G_2 \cap (x_0^{-1}G_1x_o))$to C Moreover if t and z correspond under the map and $\eta \in G_2$ then $[t]\eta$ and $[z]\eta$ do also. Finally the functions $\|f\|^2$ and $\|\tilde{f}\|^2$ define function on C and G_2 / $(G_2 \cap (x_0^{-1}G_lx_0))$ respectively which correspond under this map.

If we use this same map to transfer the measure μ_c in C over to the homogeneous space $G_2/(G_2 \cap (x_0^{-1}G_1x_o)$ we will get a quasi invariant measure v There such that $\int\|f(x)\|^2\, d\mu(z) = \int\|\tilde{f}(x)\|^2\, dv(z')$. Thus $f \to f'$ sets up the unitary transformation demanded by the conclusion of the lemma.

Let G_1 and G_2 be closed subgroups of the separable locally compact group $\mathfrak{G}$.

We shall say that G_1 and G_2 are discretely related if there exists a subset of $\mathfrak{G}$ whose complement has Haar measure zero and which is itself the union of countably many $G_1{:}\,G_2$ double cosets. Since the double cosets $G_1{:}\,G_2$ are in an obvious natural one-to-one correspondence with the orbits in $\mathfrak{G}/G_1$ under G_2 the discussion in the preceding and Lemma (1.2.17).

Theorem (1.2.18)[11]: Let U^{L} be the representation of $\mathfrak{G}$ induced by the representation L of the closed subgroup G_1 of $\mathfrak{G}$. Let G_2 be a second closed subgroup of $\mathfrak{G}$ and suppose that G_1 and G_2 are discretely related. For each $x \in \mathfrak{G}$ consider the subgroup $G_2 \cap (x^{-1}G_1x)\ of\ G_2$ and let xV denote the representation of G_2 induced by the representation $\eta \to L_{x\eta x^1}$ of this subgroup.Then xV is determined to within unitary equivalence by the double coset $G_1xG_2 = D(x)$ to which x belongs and we may write $_{\mathrm{D}}\mathrm{V} = {}_{\mathrm{x}}\mathrm{V}$ where D = D(x). Finally U^{L} restricted to G_2 is the direct sum of the $_{\mathrm{D}}\mathrm{V}$ over those $G_1{:}\,G_2$ double cosets D which are not of measure zero.

As a fairly easy Corollary of this Theorem combined with Theorem(1.2.15) we get .

Theorem (1.2.19)[11]: Let G_1 and G_2 be discretely related closed subgroups of $\mathfrak{G}$and let L M be representations of G_1 and G_2 respectively. For each $x,y \in \mathfrak{G}\, X\, \mathfrak{G}$ consider the representations $s \to L_{xsx^{-1}}$ and $s \to M_{ysy^{-1}}$ of the subgroup $(x^{-1}G_1x) \cap (y^{-1}G_2y)$ of $\mathfrak{G}$. Let us denote their Kronecker product by $N^{x,y}$ and form the induced representation $U^{Nx,y}$ of $\mathfrak{G}$. Then $U^{Nx,y}$, is determined to within unitary equivalence by the double coset $G_1\, x_y^{-1}\, G_2$ to which x_y^{-1} belongs and the direct sum of the $U^{Nx,y}$ over those double cosets which are not of measure zero, is unitary equivalent to the Kronecker Product $U^L \otimes U^M\ of\ U^L$ and U^M.

Proof: $U^L \otimes U^M$ Is the representation of $\mathfrak{G}$ obtained from the representation $U^L \otimes U^M\ of\ \mathfrak{G} \times$by restriction to the isomorphic replica $\tilde{\mathfrak{G}}$ of $\mathfrak{G}$ consisting of all $x,y \in \mathfrak{G}\, X\, \mathfrak{G}\ with\ x = y$. Moreover by Theorem(1.2.15) , $U^L \times U^M$is unitary equivalent to $U^{\mathrm{L}\times\mathrm{M}}$ where $L \times M$ is of course a representation of $G_1 \times G_2$.

Thus we have to do with the restriction of an induced representation to a subgroup and may try to apply Theorem(1.2.18). An easy computation shows that the mapping $x,y \to xy^{-1}\ of\ \mathfrak{G} \times \mathfrak{G}\ on\ \mathfrak{G}$ sets up a one-to-one correspondence between the double cosets $(G_1 \times G_2){:}\,\tilde{\mathfrak{E}}\ of\ \mathfrak{G} \times \mathfrak{G}$ and the double cosets $G_1{:}\,G_2\ of\ \mathfrak{G}$ in which $(G_1 \times G_2)\,(x,y)\,\tilde{\mathfrak{G}}$ corresponds to $G_1xy^{-1}G_2$. Furthermore in this mapping double cosets of measure zero correspond to double cosets of Measure zero.

Indeed x_1,y_1 and x_2,y_2 go into the same point of $\mathfrak{G}$ if and only if they belong to the same left $\tilde{\mathfrak{G}}$ coset. Thus a one-to-one mapping of $\mathfrak{G}$ onto the left coset space $(\mathfrak{G} \times \mathfrak{G})//\tilde{\mathfrak{G}}$ is induced. By Lemma (1.2.3)(which is of course equally true for left coset spaces) a double coset in $\mathfrak{G} \times \mathfrak{G}$ is of measure zero if and only if its image in$(\mathfrak{G} \times \mathfrak{G})//\tilde{\mathfrak{G}}$ is of Measure zero with respect to the quasi invariant measures in this coset space.

On the other hand the measure in transferred to $(\mathfrak{G} \times \mathfrak{G})//\tilde{\mathfrak{G}}$by the one-to-one mapping in question is readily seen to be quasi invariant. Thus the hypotheses of Theorem (1.2.18)are satisfied. By this Theorem $U^{L\times M}$ restricted to $\tilde{\mathfrak{G}}$ is a direct sum over the double cosets $(G_1\ X\ G_2)\ (x,y)\ \tilde{\mathfrak{G}}$ which are not of measure zero and the summand associated with the double coset containing x, y is the representation of 3 induced by the representation $s,s\ \rightarrow\ (L\ \times M)_{(x,y)(s,s)(x,y)^{-1}}$ of $\tilde{\mathfrak{G}}\ \cap\ ((x,y)^{-1}(G_1\times\ G_2)\ (x,y))$.

But it is easily verified that $\tilde{\mathfrak{G}}\ \cap\ ((x,y)^{-1}(G_1\times\ G_2)\ (x,y))$ transferred to $\mathfrak{G}$ by the natural isomorphism is the subgroup $x^{-1}G_1x\ \cap\ y^{-1}G_2y$ and that the representation $s\ ,s\ \rightarrow$ $(L\ \times M)_{(x,y)(s,s)(x,y)^{-1}}$ becomes the representation $L'\times M'$ where L' is $s\rightarrow L_{xsx}$and $M'$is $s\rightarrow M_{ysy^{-1}}$.

Let U and V be representations of the separable locally compact group $\mathfrak{E}$. A bounded linear operator T from $\mathfrak{H}(V)\ to\ \mathfrak{H}(U)$ will be called an intertwining operator for U and V if $U_xT\ =\ TV_x$.for all $x\ \in\mathfrak{G}$.

If T is in addition a Hilbert Schmidt operator it will be called a strong intertwining operator. The dimension $0,1,2,..,\infty$ of the vector space of all intertwining operators For U and V will be called the intertwining number I(U; V) of U and V. The dimension of the vector space of all strong intertwining operators will be called the strong intertwining number J(U, V) of U and V.

Let $(\mathfrak{H}(U))$, be the smallest closed subspace of $\mathfrak{H}$ (U) which contains all finite dimensional subspaces of $\mathfrak{H}(U)$ which are invariant under U. Then $(\mathfrak{H}(U))_f$itself an invariant subspace of $\mathfrak{H}(U)$. The component of U in this invariant subspace we shall call the finite discrete part of U and denote it by ^0U.

Lemma (1.2.20)[11]: Let U and V be representations of the separable locally compact group $\mathfrak{G}$. Then J(U, V) (D(U)) $=\ I(^0U,^0V)$ and this number is equal to the number of times that $U\otimes\bar{V}$ contains the identity representation as a discrete direct summand; that is the dimension of the subspace of $\mathfrak{H}(U)$ in which all U_x.act as the identity.

Proof. If $U_xT\ =\ TV_x$. then $U_xTV_x^{-1}\ = T$ which may be written $U_xT_x^*= T$ Or $\left(U\otimes\tilde{V}\right)_x(T)=T$. Since all steps are reversible the equality of J(U, V) to the dimension of the identity component of $U\otimes\tilde{V}$ is established. We now show the equality of J(U, V) and $I(^0U,^0V)$. Let T be any strong intertwining operator for U and V. Let M_1 be the orthogonal complement of the null space of T and let M_1 be the closure of the range of T. Since T is an intertwining operator it follows that $M_1$and M_2 are invariant under U and V respectively. Let A(v) $=$ (T*(T(v)*))*. Then A is a self adjoint operator in $\mathfrak{H}(V)$ which commutes with all V_x and is completely continuous. Because of the latter property it has a pure point spectrum and each non zero value occurs only a finite number of times. It follows that M_2 is a direct sum of finite dimensional invariant subspaces and a similar argument shows that the same is true of M_1. Thus $M_2\ \subseteq(\mathfrak{H}(V))_f$ and $M_2\subseteq$(&(U))f . Hence every strong intertwining operator carries $(\mathfrak{H}(V))_f$ into $(\mathfrak{H}(U))_f$fand is zero on the orthogonal complement of $(\mathfrak{H}(V))_f$ it follows at once that $J(^0U,^0V)\ =\ J(U;\ V)$. Finally it is evident that both $I(^0U,^0V)$ and $J(^0U,^0V)$are equal to $\sum_{w^{n}w^{m}w}$ where the sum is over all finite dimensional irreducible representations of $\mathfrak{G}$ which appear as components of either $^0U\ or\ ^0V$, and where $n_w\ (resp. m_w)$ is the multiplicity of occurrence of W in ^0U (resp. ^0V).

Lemma (1.2.21)[11]: Let L be a representation of the closed subgroup G of the separable locally compact group $\mathfrak{G}$.
Then the number of times that U^{L} contains the identity as a discrete direct summand is equal to the number of times that L contains the identity as a discrete direct summand provided that $\mathfrak{G}/G$ admits a finite invariant measure. If $\mathfrak{G}/G$ does not admit a finite invariant measure thenU^{L} does not contain the identity as a discrete direct summand.
Proof: Suppose first that $\mathfrak{G}/G$ admits a finite invariant measure $\mu. Let\ f$ be any member of $,\mathfrak{H}(\mu_{U^L})$ such that f(xs) = f(x) almost everywhere in x for each s. Then $f(x)$ is almost everywhere equal to a certain constant vector $v. Since\ \ f(\xi x) = L_\xi(f(x))$ for all x in $\mathfrak{E}$ and all ξ in G it follows that $v = L_\xi(v)$ for all $\xi \in G$. Conversely let v be any member of $\mathfrak{H}(L)$ such that $v = L_\xi(v)$ for all $\xi \in G$ and let $f(x) = v$ for all $x \in \mathfrak{G}$.
Since $,\mu$ is finite it follows at once that $f \in \mathfrak{H}(\mu_{U^L})$ and that $U_s(f) = f$ for all $s\ in\ \mathfrak{G}$. Thus $v \to f_v$ where $v \in \mathfrak{H}(L)$ and $f_v(x)$ sets up a one-to one linear map of the identity component of $\mathfrak{H}(L)$onto the identity component of , $\mathfrak{H}(\mathrm{U}^L)$ and the first part of the lemma is showd.
Now let $,\mu$be any quasi invariant measure in $\mathfrak{G}/G$ and let ρ be an associated ρ -function. Suppose that U^{L} contains the identity a non zero number of times and that f Is a non trivial member of the corresponding subspace of $\mathfrak{H}(,\mu_{U^L})$. Then $f\sqrt{(xs)\,\rho(xs)/\rho(x)} = f(x)$ for all s for almost all x. Thus $f(x)\sqrt{\rho(x)}$ is almost everywhere equal to some fixed vector v. Thus for each $\xi \in G$ and almost all $x\,v/\rho(\xi x) = L_\xi(v)/\rho(x)$. It follows that $\rho(\xi x) = \rho(x)$ for all $\xi$and x and hence that $\rho$is constant on the right G cosets. Now $\|f\|^2 = \|v\|^2\int(1/p(x))\,d\mu(z)$. Thus the measure whose Radon Nikodym derivative with respect to $\mu\ is\ {}^1/_{\rho'}$(where $\rho'$is the function on $\mathfrak{G}/G$ defined by ρ) is a finite measure. Moreover it is easily seen to be invariant.

With the aid of these two lemmas and Theorem(1.2.19) we may show the discrete case of the third main Theorem.
Theorem (1.2.22)[11]: Let $\mathfrak{G}, G_1, G_2, L$ and M be as in Theorem(1.2.19) .For each x and y in $\mathfrak{G}$ consider the representations $s \to L_{xsx^{-1}}$, and $s \to M_{ysy^{-1}}$, of $(x^{-1}G_1x)\cap(y^{-1}G_2y)$ and let $J(L;\ M;\ x, y)$ denote their strong intertwining number. Then $J(L, M, x, y)$ depends only upon the double coset $D = D(x, y) = G_1xy^{-1}G_2$ to which xy^{-1} belongs so that we may write $J(L, M, D)$. Moreover whether or not $(x^{-1}G_1x) \cap (y^{-1}G_2y)$ is such that $\mathfrak{G}/((x^{-1}G_1x) \cap (y^{-1}G_2y))$ admits a finite invariant measure depends only on this double coset. Let be the set of all double cosets for which a finite invariant measure does exist and which are not of measure zero. Then

$$\sum_{D\in\mathfrak{D}f} J(L, M, D) = J(U^L, U^M)$$

Proof: By Lemma (1.2.21), $J(U^L, U^M)$ is equal to the number of times that $U^L\otimes\overline{U^M} = U^L \otimes U^{\overline{M}}$ contains the identity. By Theorem(1.2.19) , $U^L \otimes U^{\overline{M}}$ is a direct sum over the double cosets of positive measure of certain induced representations U^{D^N} .Hence $J(U^L, U^M)$is the sum over these double cosets of the number of times that U^{D^N}contains the identity. By Lemma (1.2.21) agiven deon tributes to this sum only if $D \in \mathfrak{D}_f$ and then its contribution is

the number of times that the Kronecker product D^{N} contains the identity .But again by lemma (1.2.20) the number of times that D^{N} contains the identity is exactly J(L, M, D). This completes the proof.
If in Theorem(1.2.20) we let $G_1 = G$ and $G_2 = \mathfrak{G}$ then there is only one $G_1: G_2$double coset and we may take x = y = e.
The Theorem then reduces to the following generalization of the Frobenius Reciprocity Theorem.
Theorem (1.2.23)[11]: Let L be an irreducible representation of the closed subgroup G of the separable locally compact group $\mathfrak{G}$ and let M be any finite dimensional irreducible representation of $\mathfrak{G}$. Then if $\mathfrak{G}/G$ admits a finite invariant measure, U^{L} contains M as a discrete direct summand exactly as many times as the restriction of M to G contains L as a discrete direct summand. If $\mathfrak{G}/G$ does not admit a finite invariant measure then U^{L} contains no finite dimensional discrete direct summands.
Corollary (1.2.24)[11]:. If $\mathfrak{G}$ is not compact then its regular representation contains no finite dimensional discrete direct summands.
Theorem (1.2.21) for compact (not necessarily separable) groups has been showed by Weil [42]. F. I. Mautner informs me that he has found a somewhat different generalization of the Frobenius reciprocity Theorem than Theorem (1.2.21) for the case in which G is compact.
In order to rid the results of Part II of the rather severe discreteness restriction there imposed we need the notion of a direct integral or continuous direct sum of Hilbert spaces. Such a notion has been developed by von Neumann [36] and in another form by Godement [21], [22].

Mautner [31], [32], [33], and Godement [22] have applied this notion to the decomposition of unitary representations of locally compact groups.
For our own use we have written up the theory in a form differing slightly but not essentially from that of both von Neumann and Godement.

We shall sketch our form of the theory so as to have its principal results available for later use.

We shall call the members of this field the Borel sets in Y. Let μ be a completely additive measure defined on all Borel sets and σ-finite in the sense that Y is a union of countably many Borel sets of finite measure.
The members of the smallest σ -field containing all Borel sets and all subsets of Borel sets of measure zero will be called the measureable subsets of Y . The measure μ of course has a unique extension to this larger - σ field.

We shall call the system of objects just described a Borel measure space. Now suppose that there is given for each y ϵ Y a finite or infinite dimensional separable Hilbert space $\mathfrak{H}_y$. Let $\mathfrak{J}$ denote the set of all functions f from Y to $U_{y \in Y}\mathfrak{H}_y$ such that $f(y) \in \mathfrak{H}_y$ for all y and such that $y \to \| f(y)\|^2$ is, μ summable on Y. We shall call a subset $X\ of\ \mathfrak{F}$ linear if it is closed under the formation of finite linear combinations of its members. It is easy to see that $y \to (f(y), g(y))$ is μ summable whenever f and g are members of a linear X and that X becomes a (possibly incomplete) Hilbert space under the scalar product $(f: g) = \int (f(y), g(y)\ d\mu(y)$ when functions which are equal almost everywhere are identified. If X is maximal linear in the sense that it is contained in no properly larger linear X' then a slight and obvious modification of the standard proof of the completeness of the space of square summable functions shows that X is complete and hence is a Hilbert space.

Let X be a linear subset of $\mathfrak{I}$. We shall say that X is pervasive if it contains a countable family of elements $f_1, f_2, \ldots$ such that for almost every y in Y the

Sequence of elements $f_1(y), f_2(y), \ldots$ has $\mathfrak{H}_y$ as its closed linear span. Modulo automorphism of the $\mathfrak{H}_y$ there is at most one pervasive maximal linear X. We have

Lemma (1.2.25)[11]: Let$X_1$and X_2 be two pervasive maximal linear subsets of $\mathfrak{F}$. Then for each $y \in Y$ there exists a unitary transformation $U_y\ of\ \mathfrak{H}_y$ onto itself such that $f \in X_1$ if and only if $g \in X_2$ where $g(y) = U_y\left(f(y)\right)$ for almost all $y \in Y$.

Proof: By an orthogonalization process it is possible to replace the given pervading sequences $f_1, f_2, \ldots, 9_1, 9_2, \ldots$. for X_1 and X_2 respectively by new sequences $f_1', f_2', \ldots$ and $g_1', g_2', \ldots$ having the following properties: (a) each $f_1'\ (resp. g_i')$ is the product of a complex valued measurable function and a member of $X_1\ (resp. X_2)$, (b) for almost all $y, f'(y) = 0$ implies $f_{i+1}'(y) = 0$ and $g_i'(y) = 0$ implies $g_{i+1}'(y) = 0, (c)$ for almost all $y\ the\ f_i'(y)$ which are not zero form a complete orthonormal set in $\mathfrak{H}_y$ and the same is true of the $g_i'(y). U_y$ is then defined so that for almost all $y, U_y(f_i'(y)) = g_i'(y)$.

There is no reason for supposing that a pervasive maximal linear X will exist in general and indeed it need not.

On the other hand if all of the $\mathfrak{H}_y$ have the same dimension one can be constructed as follows. Simply map all of the $\mathfrak{H}_y$ onto a fixed representative $\mathfrak{H}_0$ and consider the set of all functions f from Y to $\mathfrak{H}_0$ such that $(f(y), v)$ is measurable in y for each v in $\mathfrak{H}_0$ and $(f(y), f(y))$ is summable. More generally suppose that for $n = \infty, 1, 2, \ldots$ the y with $\mathfrak{H}_y$, of dimension n form a measurable set Y_{n}. For each n let us map the $\mathfrak{H}_y$ of this dimension on a fixed representative$\mathfrak{H}_n$. Then we may obtain a pervasive maximal linear X as follows. Let X be the set of all functions from $Y\ to\ U\mathfrak{H}_n$ such that $f(y) \in \mathfrak{H}_n$ for all $y \in Y_n$ such that $(f(y), v)$ is measurable on Y_n for each $v \in \mathfrak{H}_n$ and such that $\int (f(y), f(y)\ d\mu(y) < \infty$. Conversely it follows from Lemma(1.2.1) and the argument used to show it that this is the general situation.

If a pervasive maximall linear X exists then Y_n is measurable for each n and X can be defined in the manner just described. In the special case in which Y is a finite or countable set and μ is never zero it is clear, because of the measurability of all functions, that there is only one pervasive maximal linear X and that is $\mathfrak{I}$ itself.

In general however there will be, if any, many different but equivalent such $X's$ corresponding to the many ways of mapping the $\mathfrak{H}_y$ of a given dimension onto a representative-which are not derivable from one another in a "measurable fashion". We shall refer to each pervasive maximal linear X as a direct integral of the $\mathfrak{H}_y$with respect to μ. In applications there is often a "natural" choice of X.

In particular there will often be given a pervasive linear subset of $\mathfrak{I}$ and it is not difficult to show that every pervasive linear subset of $\mathfrak{I}$ is contained in a unique (pervasive) maximal linear subset of $\mathfrak{I}$.

Let X be a direct integral of the $\mathfrak{H}_y$, and suppose that we are given a bounded operator ${}^{y}T$ in each $\mathfrak{H}_y$. If $('T(f(y)), g(y))$ is measurable in y for each f and g in X and if $\|'T\|$ is

bounded then for each f in $X, y \rightarrow' \mathrm{T}(\mathrm{f(y)}) = \mathrm{f_T(y)}$ will be in X also and $f \rightarrow \mathrm{f_T}$ will define a bounded linear operator T of X into itself.
We shall call T the direct integral $\int 'T\ d\mu(y)$ of the $^{\mathrm{V}}$T with respect to X. It is not difficult to show that this notion has the expected elementary properties; that is

$$\int v_{T^*} d\mu(y) = (v_{T^*} d\mu(y))^*, \int (v_{T^v} S)\, d\mu(y) = \left(\int v_T d\mu(y)\right)\left(\int v_S d\mu(y)\right)$$

and soon.

Now suppose that there Is given in each $\mathfrak{H}_y$ a representation $v_U(x \rightarrow v_{U_x})$) of a fixed separable locally compact group $\mathfrak{G}$. We shall say that the mapping $y \rightarrow v_U$ is measurable with respect to a maximal pervasive linear X if for each fixed $x \in \mathfrak{G}$ the direct integral $V_x = \int v_{U_x} d\mu\ (y)$ of the v_{U_x} with respect to X exists in the sense described in the preceding paragraph.

We shall say that $y \rightarrow v_U$ is measurable if it is measurable with respect to X for some X. Let now y→ v_U be measurable (with respect to X say).
It follows from the elementary properties of direct integrals of operators that
Each V_{x} is unitary and that $x \rightarrow V_{\mathrm{x}}$ has the algebraic properties of a representation. In order to show that $x \rightarrow V_{\mathrm{x}}$ is actually a representation we have need of the following measure theoretic lemma.

Lemma (1.2.26)[11]: Let Y be a Borel measure space and let v be a Borel measure in a separable locally compact metric space $\mathfrak{M}$.
Let f be a complex valued function defined on $\mathfrak{M} \times Y$ which for each fixed y in Y is continuous on $\mathfrak{M}$ and for each fixed x in $\mathfrak{M}$ is measurable on Y.
Then f is a measurable function on the product space $\mathfrak{M} \times Y$.

Proof: There is clearly no loss in generality in upposing that $\mathfrak{M}$ is compact. For each $n = 1, 2, \ldots$ we may Write $\mathfrak{M}$ as a disjoint union of Borel sets each of diameter less than $1 / n$: $\mathfrak{M} = m_1^n \cup m_2^n \cup \ldots m_{j_n}^n$ Choose a point x_j^n in each m_j^n and let $f_{\mathrm{n}}\ (x, y) = f(x_j^n, y)$ for all x in m_j^n and all y in Y. Then it is obvious that f_n, is measurable in both variables and a very easy argument shows that for all x, and y, $lim_{n\rightarrow\infty} f_n(x,\ U\ldots y) = f(x, y)$.

Now let f and g be arbitrary members of X and note that $(V_x(f)\colon g) = \int (v_{U_x}(f(y)), g(y))\ d\mu(y)$. Since $v_{\mathrm{U}}$is a representation the integrand is continuous in x for each y and by definition it is measurable in y for each x. Thus by Lemma (1.2.25) this integrand is measurable in both variables. Applying the Fubini Theorem we conclude that $(V_x(f)\colon g)$ is measurable in x so that V is in fact a representation. It is readily verified that the following operations will carry V into a representation which is unitary equivalent to itself: (a) Change of y_{U} on a set of measure zero. (b) Replacement of X by any other maximal pervasive linear X' with respect to which $y - v_U$ is measurable. (c) Replacement, of each v_U by a unitary equivalent representation. (d) Replacement of ,μ by any other measure with the same null sets. We shall call V the direct integral of the v_U with respect to μ. $V = \int v_U\ U d\mu(y)$.

Theorem (1.2.28)[11]: Let G be a closed subgroup of the separable locally compact group $\mathfrak{G}$. Let M be a representation of G which is a direct integral over a Borel measure space Y, μ_1 of representation$s^y\ L$; $M = f\ 'L\ d\mu(y)$. Then $y \rightarrow U^{Y_L}$ Is measurable and $\int U^{Y_L}\ d\mu(y)$ is unitary equivalent to U^{M}.

In any event M will be a discrete direct sum of representations each of which is a direct integral of ${}^{v}L's$ having a common dimension. Thus we need only consider the case in which all of the $\mathfrak{H}({}^{v}L)$ have the same dimension.

We may assume without loss of generality that there is a single Hilbert space $\mathfrak{H}_1$ in which all${}^{v}L$ act and that $\mathfrak{H}(M)$ is the set of all functions from $Y\ to\ \mathfrak{H}$ such that $(f(y), w)$ is measurable for all $w \in \mathfrak{H}_1$ $\int \left(f\ (y), f(y)\right)^2 dj\mu(y) < \infty$.

Now let $C_{\mathfrak{H}_1}$, denote the set of all functions from $\mathfrak{G} to\ \mathfrak{H}_1$ which are continuous and have compact support.

For each $g \in C_{\mathfrak{H}_1}$ and each $y \in Y$ we may define $g_y^0(x)$ as in Lemma (1.2.8)using $L\ for\ L$. We shall write $g_y^0(x) = \hat{g}(x, y)$. By Lemma (1.2.8), $\hat{g}(x, y)$ is for each fixed y a member of , $\mathfrak{H}(\mu_{U^y}L)$ where ,μ is any quasi invariant measure in $\mathfrak{G}/G$ Now let $\mathfrak{S}$ be the set of all functions from $\mathfrak{G} \times Y$ to $\mathfrak{H}_1$ of the form $\emptyset_1\ \hat{g}_2 + \dots . + \emptyset_n \hat{g}_n)$ where each $g_j \in C_{\mathfrak{H}_1}$, and each $\emptyset_j$ is a bounded measurable complex valued function of y alone which vanishes outside of a set of finite measure. Let r be any member of $\mathfrak{S}$. For each fixed $x \in \mathfrak{G}, r(x, y)$ is a function from $Y\ to\ , \mathfrak{H}_1$.

It is not difficult to show that it is weakly measurable, bounded and zero outside of a set of finite measure. Thus it is a member of $\mathfrak{H}(M)$.We denote this member of $\mathfrak{H}(M)\ by\ r'_x$. There are also no difficulties in showing that the maping $x \to r'_x$ is continuous from $\mathfrak{G}\ to\ \mathfrak{H}(M)$.and is indeed a member of $\mathfrak{H}(U^M)$. Finally by Lemma(1.2.11) the set of all members of $\mathfrak{H}(U^M)$ of the form $x \to r'_x$ for $r \in \mathfrak{S}$ is dense in $\mathfrak{H}(U^M)$. On the other hand the members of $\mathfrak{S}$ may be looked at in another way. For each fixed $y\ in\ Y, r(x, y)$ as a function of x is a member of $\mathfrak{H}(x_{U^2}L)$. Call it r''_y! The mapping $y \to r''_y$ is thus a function in the class $\mathfrak{I}$ use in constructing direct integrals of the spaces $\mathfrak{H}(x_{U^2}L)$. The set R of all members of $\mathfrak{I}$ of the form $y \to r''_y$ for r in e is readily seen to be linear and pervasive. Moreover it is closed with respect to multiplication by bounded measurable functions of y which vanish outside sets of finite measure.

By a result in direct integral theory then R is dense in the unique maximal pervasive linear subset of $\mathfrak{I}$ containing R. Call this maximal set X. It is easy to see that $y \to U^y L$ is measurable with respect to X. We take X then as $\mathfrak{H}(\int U^y L\ d\mu(y))$.

It is an immediate consequence of the Fubini Theorem that the member of $\mathfrak{H}(U^M)$ defined by $r \in \mathfrak{S}$ has the same norm as the member of X defined by r. Thus we have a norm preserving linear map of a dense subspace of $\mathfrak{H}(U^M)$on the dense subspace $R\ of\ X = \mathfrak{H}(\int U^y L\ d\mu(y))$. This extends to a unitary map of one space on the other which can be seen without difficulty to set up the desired unitary equivalence.

Corollary (1.2.28)[11]: Let the regular representation of G be decomposed as a direct integral of representations y_L . Then the regular representation of $\mathfrak{G}$ is a direct integral of the representations U^{Y_L}.

The decomposition of the regular representation of a group defined by a decomposition of the regular representation of a subgroup has been noted by Godement [9] for the case in which the subgroup is Abelian and the decomposition is into one dimensional parts.

Let $\mathfrak{M}$ be a separable locally compact space and let ,μ be a finite measure in $\mathfrak{M}$. Let there be given an equivalence relation in $\mathfrak{M}$. Let the equivalence classes form a set Y and for each $x \in \mathfrak{M}\, let\, r(x) \in Y$ denote the equivalence class to which x belongs. Following Rohlin [38] we shall say that the equivalence relation is measurable if there exists a countable family E_1, E_2, ...of subsets of Y such that $r^{-1}(E_i)$ is measurable for each i and such that each point y of Y is the intersection of the E_{i} which contain it. We shall need a lemma asserting that ,μ may be "decomposed" as an integral over Y of measures μ_y concentrated in the various equivalence classes.

Lemma (1.2.29)[11]: Let $\tilde{\mu}$ be the measure in Y such that $E \subseteq Y$ is measurable if and only if $r^{-1}(E)$ is μ measurable and such that $\tilde{\mu}(E) = \mu(r^{-1}(E))$.

Then for each y in Y there exists a finite Borel measure $\mu_y\, in\, \mathfrak{M}$ such that $\mu_y(\mathfrak{M} - r^{-1}(\{y\})) = 0$ and $\int f(y) \int g(x) d\mu(x)\, d\tilde{}(y = \int f(r(x)) g(x)\, d\mu\,(x)$ whenever $f \in \mathfrak{L}^1\,(Y, \tilde{\mu})$ and g is bounded and measurable on $\mathfrak{M}$.

We shall not stop to give a proof of this lemma. In one form or another it has been showd in a number of places.

See for example von Neumann [34], Halmos [23], [24], Dieudonne [13], and Rohlin [38]. The formulation given here has been influenced by conversations on the subject with R. Go dement and a reading of a joint manuscript of Godement and N. Bourbaki. The Godement Bourbaki treatment will presumably appear in a subsequent volume of N. Bourbaki's well known treatise.

We shall apply Lemma (1.2.29) when $\mathfrak{M}$ is the homogeneous space $\mathfrak{G}/G$ and μ is a quasi invariant measure in $\mathfrak{G}/G$. We shall need to know that the μ_y are also quasi invariant and devote to a proof of this fact.

Lemma (1.2.30)[11]: Let μ_1 and μ_2 be Borel measures in the separable locally compact spaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Let r_1, Y_1 and r_2, Y_2 define measurable equivalence relations in $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively.

Let $Y = Y_1\, X\, Y_2$ and let r, where $r(x_1 \times x_2) = r(x_1), r(x_2)$ be the product equivalence relation in $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Then r is measurable and in the decomposition of $\mu_1 \times \mu_2$ by Lemma (1.2.29) we may take $(\mu_1)_y \times (\mu)_2\; for\; (\mu_1 \times \mu_2)_y$.

Proof: The proof results from writing down the defining equation of $(\mu_1 \times \mu_2)_y$ and making a few obvious manipulations.

Lemma (1.2.31)[11]: Let μ, r, Y and $\mathfrak{M}$ be as in Lemma(1.2.29) and let t be a homeomorphism of $\mathfrak{M}$with itself such that $r([x]t) = r(x)\, for\, all\, x\, in\, \mathfrak{M}$. Let $\mu^t(E) = \mu([E]t)$. Then in decomposing μ^t by Lemma(1.2.29) $(\mu^t)_y$, may be taken to be $\left(\mu_y\right)^t$

Lemma (1.2.32)[11]: Let μ, r, Y Let and $\mathfrak{M}$ be as in Lemma (1.2.29) and let k be a non negative function on $\mathfrak{M}$ which is μ summable. v be the measure whose Radon-Nikodym derivative with respect to ,μ is k. Then $\tilde{v}$ is absolutely continuous with respect to $\check{\mu}$the Radon-Nikodym derivative being λ say.

Moreover in the decomposition of $v\, v_y$ may be taken to be that measure, absolutely continuous with respect to μ_y whose Radon-Nikodym derivative is zero or $x \to k(x)/\lambda(y)$ depending upon whether or not $\lambda(y)$ is zero.

Proof: It follows by an easy argument from Lemma (1.2.29) that for all $\tilde{v}$ measurable sets $A, \tilde{v}(A) = f_A \int k(x)\; d_{\mu_y}(x)\, d\tilde{\mu}(y)$. Thus $\tilde{v}$ is absolutely continuous and we may take $\lambda(y) = \int k(x)\; d_{\mu_y}(x)$. The defining equations of μ_y and v_y lead at once to the equation

$$\int f(y)\lambda(y) \int g(x)dv_y(x)d\tilde{\mu}(y) = \int f(y)\int g(x)k(x)d\mu_y(x)\, d\tilde{\mu}(y).$$

Thus for $\tilde{\mu}$ almost all $y, \lambda(y)$ $\int g(x)dv_y(x) = \int g(x)k(x)d_{\mu_y}(x)d\tilde{\mu}(x)$. now $\lambda(y) > 0$ for almost oll y . Thus for almost $\tilde{v}$ all y

$\int g(x)dv_y(x) = \left(1 / (\lambda(y)\right) \int g(x)k(x)d_{\mu_y}(x)$.

and the truth of the lemma follows.

Lemma (1.2.33)[11]: Let $\mathfrak{M}$, r, Y and μ be as in Lemma (1.2.29) and let the separable locally compact group $\mathfrak{G}$act on $\mathfrak{M}$ in such a manner that: (a) x → [x]z is a homeomorphism t_z for each z ϵ $\mathfrak{G}$, (b) z → $t_z$is a homomorphism of $\mathfrak{G}$ into the group of homeomorphisms of $\mathfrak{M}$ onto itself. (c) $[x]z$ is continuous in both variables together, (d) $r([x]z) = r(x)$ for all z in $\mathfrak{G}$ and all $x \in \mathfrak{M}$.

(e) is quasi invariant under the action of $\mathfrak{G}$. Then in the decomposition of μ almost all of the .μ_y are also quasi invariant under $\mathfrak{G}$.

Proof: Let v be any finite quasi invariant measure in $\mathfrak{G}$. Let $\mathfrak{M}_0 = \mathfrak{M}\, X\mathfrak{G}$ and $Y_0 = Y\, X\, \mathfrak{G}$. For all $x, z \in \mathfrak{M}_0\ let\ r_0(x,z) = r(x), z$ and let $(x,z)t = [x]z, z$.

Then r_0 is a measurable equivalence relation and t is a self homeomorphism of $\mathfrak{M}_0$ No. Since the hypotheses of Lemma (1.2.31) are clearly satisfied we may, in decomposing by Lemma (1.2.29),choose $[(\mu \times v)^t]_{v,z}\ as\ ((\mu \times v)_{v,z})^t$. By Lemma (1.2.30), we may choose $(\mu_y \times v_z) for\ (\mu \times v)_{y,z}$ where v_z is a measure concentrated in the point z and such that $v_z(\{z\}) = 1$. On the other hand it is evident that $(\mu_y \times v_z)^t = (\mu_y)^z \times v_z$ .Thus$[(\mu \times v)^t]_{y,z}$ may be taken as $(\mu_y)^z \times v_z$ Now$(\mu \times v)^t$ is readily seen to be absolutely continuous with respect to $\times\ v$. X v because of the quasi invariance of μ. By Lemma (1.2.32) for $\tilde{\mu} \times$ v almost all pairs y, z, the measure $(\mu_y)^z \times v_z$ is absolutely continuous with respect to$(\mu_y \times v_z)$.

Thus for almost all y he measure$(\mu_y)^z$is absolutely continuous with respect to $\mu_y$for almost all z. But for fixed y the set of all z for which $(\mu_y)^z$ is absolutely continuous with respect to μ_y is closed under multiplication. But the v null sets of $\mathfrak{G}$ are just those of Haar measure zero and it is easily seen that a multiplicatively closed subset of a group cannot have a complement of Haar measure zero unless the subset is the whole group.Thus for almost all $y,\ .\mu_y$ invariant.

We are now in a position to show generalizations of Theorems (1.2.18) and (1.2.19) for the case in which the closed subgroups $G_1$and G_2 are not necessarily discretely related. On the other hand we may not allow completely general pairsG_1, G_2.They must be related in such a manner that almost all of the orbits in $\mathfrak{G}/G_1$ under the action of G_2 form the equivalence classes of a measurable equivalence relation. In any case we can of course find a countable set $E_1, E_2, ...$ of Borel unions of orbits which generates (modulo null sets) the a field of all measurable unions of orbits.

The unique equivalence relation r such that r(x) = r(y) if and only if x and y are in the same sets E_j will be measurable and this measurable equivalence relation will define a decomposition of the quasi invariant measure into smaller quasi invariant parts. This will lead in turn to a decomposition of U^L restricted to G_2.

However the equivalence classes may be unions of many orbits instead of single orbits. When this happens the components of the decomposition of U^{L,G_2} will not be associated with single double cosets and will not be identifiable as induced representations of G_2 as in Theorem (1.2.18). In fact at the present time we know little or nothing about the nature of these components. Presumably of course the theory of the decomposition of measures into ergodic parts can be used to show that all of these components are imprimitive with respect to ergodic but not necessarily transitive systems of imprimitivity. But as indicated in [28] we know very little about non transitive systems of imprimitivity. Just as we did in [28] then we restrict ourselves to the case in which this phenomenon of non transitive ergodicity does not arise.

Specifically we define the closed subgroups G_1 and G_2 of the separable locally compact group to be regularly related if there exists a sequence $E_0, E_2, E_2, \ldots$ of measurable subsets of $\mathfrak{G}$ each of which is a union of $G_1 : G_2$ double cosets such that E_0 has Haar measure zero and each double coset not in E_0 is the intersection of the E_j which contain it. Because of the correspondence between orbits of $\mathfrak{G}/G_1$, under G_2 and double $G_1 : G_2$ cosets it is clear that G_1 and G_2 are regularly related if and only if the orbits outside of a certain set of measure zero form the equivalence classes of a measurable equivalence relation.

For each $x \in \mathfrak{G}$ $let\ s(x)$ denote the $G_1 : G_2$double coset to which x belongs.

If v is any finite measure in $\mathfrak{G}$with the same null sets as Haar measure we

May define a measure$v_0\ in\ \mathfrak{D}$, the set of all $G_1 : G_2$ double cosets, by letting

$v_0(E) = v(s^{-1}(E))$ whenever E is such that $s^{-1}(E)$ is measurable. Such a measure we shall call an admissible measure in $\mathfrak{D}$. Clearly any two such have the same null sets. In terms of these notions we may state.

Theorem (1.2.34)[11]: Let U^{L} be the representation of the separable locally compact group $\mathfrak{G}$ induced by the representation L of the closed subgroup $G_1\ of\ \mathfrak{G}$. Let G_2 be a second closed subgroup of $\mathfrak{G}$ and suppose that G_1 and G_2 are regularly related.

For each x consider the subgroup $G_2 \cap (x^{-1}G_1x)\ of\ G_2$ and $let\ xV$ denote the Representation of G_2 induced by the representation $\eta \to L_{x\eta x^{-1}}$ of this subgroup.

Then xV is determined to within equivalence by the double coset $G_1xG_2 = s(x)$ to which x belongs and we may write $xV = DV\ where\ D = s(x)$. Finally U^L restricted to G_2 is a direct integral over $\mathfrak{D}$, with respect to any admissible measure in $\mathfrak{D}$, of the representations DV.

Proof: Given an admissible measure $v_0\ in\ \mathfrak{D}\ let\ v$ be the generating measure in $\mathfrak{G}$and $let\ ,\mu$ be the quasi invariant measure in $\mathfrak{G}/G_1$ defined by the equation$\mu(E) = v(h^{-1}(E))$. For each $z\ in\ \mathfrak{G}/G_1 let\ r(x) = s(h^{-1}(z))$. Then since G_1 and G_2 are regularly related r is a regular equivalence relation.

Applying Lemma (1.2.19) we find that μ is an integral of measures μD, where $D \in \mathfrak{D}$, with respect to the measure $v_0\ in\ \mathfrak{D}$. Each is concentrated in the orbit $r^{-1}(D)$ and by Lemma (1.2.33) is quasi invariant. We define μD as the $\mathfrak{H}'_c\ of\ \#\ 6$ where $C = r(D)$ and $\mathfrak{F}$ is the set of all functions f from $\mathfrak{D}$ to $U_{D\in\mathfrak{D}}\mathfrak{H}_D$ such that $f(D) \in \mathfrak{H}_D$ for all D and $\|f\|^2 is\ v_0$ summable on $\mathfrak{D}$.

We shall exhibit a natural unitary map of $\mathfrak{H}(\mu_{U^L})$ onto a pervasive maximal linear subset of $\mathfrak{F}$ and then show that this direct integral decomposition of $\mathfrak{H}(\mu_{U^L})$ yields the desired decomposition of U^{L}. Let f be any function in $\mathfrak{H}(\mu_{U^L})$. Then $\int\|f(x)\|\ a\mu(z)dv_0(D) = \int\|f(x)\|^2\ d\mu(z) < \infty$. Thus for almost all D the restriction f_D of f to D is a member of $\mathfrak{H}_D$. Moreover the function from $\mathfrak{D}\ to\ U_{D\in\mathfrak{D}}\mathfrak{H}_D$ defined as f_D or zero according to whether or not $f_D \in \mathfrak{H}_D$ is a member of $\mathfrak{F}$ with the same norm as f. Let us denote this member of $\mathfrak{F}$ by T(f). Then $(T(f))_D = f_D$ for almost all D. Let X be the range of T. X is obviously linear and complete.

Moreover it is closed under multiplication by bounded Borel functions in $\mathfrak{D}$.

Thus in order to show it maximal linear and pervasive it suffices to show it pervasive. In order to do this observe first that it is easy to show that there exists a sequence $f_1, f_2, \ldots$ of continuous members of $\mathfrak{H}(U^L)$ such that for each $x \in \mathfrak{G}\ the\ f_k(x)$ are dense in $\mathfrak{H}(L)$. Indeed the proof is carried through more or less explicitly in the latter part of the proof of Theorem (1.2.13). Now let $\mathrm{g}_1, \mathrm{g}_2, \ldots$ be a dense subset of the continuous complex valued functions with compact support on $\mathfrak{G}/G_1$, and let $\mathrm{g}'_j(x) = g_j(h(x))$. We leave it. To the reader to show that any sequence containing all of the $g_j f_i$ is pervading for X. Finally let ηbe an element of G_2 and consider the operator${}^u U_\eta^L$. It takes $f \in \mathfrak{H}(\mu_{U^L})$into the function $x \to f(x\eta)\sqrt{\rho(x\eta)/p(x)}$ Where ρ is a p-function associated with $,\mu$. Thus $T^\mu U_\eta^L T^{-1}$ is the operator defined in X by the family of operators $D \to^D A_\eta$ where $({}^D A_\eta f_D)(x) = f_D(x\eta)\sqrt{\rho(x\eta)/p(x)}$.

Let $({}^D B_\eta f_D)(x) = f_D(x\eta)\lambda_D(h(x),\eta)$ where λ_D is a λ-function associated with the measureμ_D . Since $,U_\eta^L$ is unitary it follows that DA_η is unitary for almost all D and hence that DA_η is for almost all D the same as DB_η-It follows at once that the representation μ_{U^L} is unitary equivalent to the direct integral of the Representations $\eta \to$ DB_η and by Lemma(1.2.17) that each is unitary equivalent to $xV\ for\ all\ x \in D$.Thus the Theorem is showd.

Theorem (1.2.35)[11]: Let G_1 and G_2 be regularly related closed subgroups of the separable locally compact group $\mathfrak{G}$ and let L and M be representations of G_1 and G_2 respectively. For each $x, y \in \mathfrak{G} \times \mathfrak{G}$ consider the representations $s \to L_{XSX^{-1}}$ and $s \to M_{ysy^{-1}}$ of the subgroup $(x^{-1}G_1x) \cap (y^{-1}G_1y)\ of\ \mathfrak{G}$. Let us denote their Kronecker product by $N^{x,y}$ and form the induced representation $U^{Nx,y}\ of\ \mathfrak{G}$. Then $U^{Nx,y}$ is determined to within unitary equivalence by the double coset in question. Finally $U^L \otimes U^m$ is unitary equivalent to the direct integral of the U^{D} with Respect to any admissible measure in the set $\mathfrak{D}$ of double G_1:G_2 cosets.

Proof: The deduction of Theorem (1.2.35) from Theorem (1.2.34) is almost exactly the same as the deduction of Theorem (1.2.19)from Theorem (1.2.18).

We turn now to our generalization of Theorem(1.2.22) deriving a formula for $J(U^L;\ U^M)$ in the case in which G_1 and G_2 are only assumed to be regularly related. Just as before we base the derivation on an analysis of the Kronecker product of U^L and $U^{\bar{M}}$. However there are some important differences which necessitate a more elaborate argument. Let us introduce the notation $n_1(U)$ to denote the number of times that The representation U contains the identity representation as a discrete direct summatid.

Then $J(U^L, U^M) = n_1(U^L \otimes U^{\bar{M}}). Let\ D_1, D_2, ...$ denote the $G_1: G_2$ double cosets in $\mathfrak{G}$, if any exist, which are of measure different from zero and let $\mathfrak{D}'$ denote the set of all $G_1: G_2$ double cosets which are of measure zero. Then by Theorem (1.2.35) we have $n_1(U^L \otimes U^{\bar{M}}) = \sum_i n_I(UD_j) + n_I(\int_{\mathfrak{D}'} U^D\, dv(D))$ where v is any admissible measure in $\mathfrak{D}$.
Now $n_1(U^D)$ may be computed for all D just as it was in the proof of Theorem (1.2.20). Thus we are reduced to finding out how $n_I(\int_{\mathfrak{D}'} U^D\, dv(D))$ depends upon the $n_1(U^D)$.

Lemma (1.2.36)[11]: Let $U = \int y_{Udv(y)}$ be a direct integral of the representations y_{U} of the separable locally compact group $\mathfrak{G}$. Let the Borel measure space Y, v be free of atoms. Then the set of all y for which $n_1\ ('U) > 0$ is measurable.
If this set is of measure zero then $n_1(U) = 0$. If this set is of measure greater than zero then $n_1(U) = \infty$.

Proof: We remark first that we need only consider the case in which all $\mathfrak{H}('U)$ have the same dimension. Thus as in the proof of Theorem (1.2.27) we may suppose that all of the v_{U} act in a fixed Hilbert space, $\mathfrak{H}_0$ say, and that $\mathfrak{H}(U)$ is the set of all square summable weakly measurable functions from Y to $\mathfrak{H}_0$. For each $y \in Y$ let $\mathfrak{M}_y$ denote the maximal subspace of $\mathfrak{H}_0$ on which v_{U} is the identity and let y_{E} denote the projection on $\mathfrak{M}_y$.
If we can show that $(yE(v), w)$ is measurable in y for all v and $w\ in\ \mathfrak{H}_0$ the truth of the lemma will follow. Indeed let $\phi_1, \phi_2, ...$be a complete orthonormal basis for $\mathfrak{H}_0$ and suppose that the measurability in question has been established.
Then $n_1('U) = 0$ if and only if $YE = 0$ that is if and only if $(YE(\phi_i), \phi_i) = 0$ for all i and j. Thus the set where $n_i(YU) = 0$ is the intersection of countably many measurable sets and hence is measurable itself.
Suppose that the set where $n_1('U) > 0$ has measure zero. Let f be any member of the subspace of $\mathfrak{H}(U)$ on which U is the identity. Then for each $x\ in\ \mathfrak{G}, U_x(f(y)) = f(y)$ for almost all y in Y.
Using the separability of $\mathfrak{G}$ and the continuity of $YU_x\ in\ x$ we see that for almost all $y, U_x(f(y)) = f(y) = f(y)$ for all x. Thus $E(f(y)) = f(y)$ for almost all y. But YE = 0 for almost all y. Thus $f(y) = 0.$ for almost all y. Thus $n_1(U) = 0$. Suppose now that $n_1(U) > 0$ on a set of positive measure. Since Y is atom free and countably generated there exist countably many disjoint measurable sets $Y_1, Y_2, ...$ each of positive measure and each consisting entirely of points y for which $n_1(U) > 0. Let^y\ E_i = E$ for $y \in Y_i$ and let $E_i = 0\ for\ y \notin Y_i$. Then $f \to g\ where\ g(y) = vE_i(f(y))$ defines . a non zero projection E_i for each $i = 1, 2, ...$ Since the ranges of these projections are linearly independent and since U is the identity on each it follows that $n_1(U) = \infty$. To show that $(^yE(v), w)$ is indeed measurable and thus complete the proof of The lemma we proceed as follows. First choose $s_1, s_2, ...\ dense\ in\ \mathfrak{G}$.
For each choice of $y \in Y, v$ and s_{j} it follows from the mean ergodic Theorem that $(v +^y U_{sj}(v) + U_{sj}^2(v) + ...^y U_{sj}^n(v))/(n + 1)$ converges with increasing n to^y E_{sj}(v) where^y E_{sj} is the projection on the one space of the single operator U_{sj} . It follows at once that $(\mathrm{E}_{sj}(v), w)$ is measurable in y for each j, v and w. Now it is easy to see that^{y}E is for each y simply the projection on the intersection of the ranges of the E_{sj} Let E_n be the

projection of the of the intersection ranges of E_{s1}, $E_{s1}, \ldots E_{sn}$. It is clear that $({}^{y}E_n(v), w)$ tends to $(E(v), w)$ as n tends to ∞. Thus we need only show that $(E_n(v), w)$is measurable for each n; or more generally that if $y \to E_1$ and $y \to E_2$ are families of projections such that $(E_1(v), w)$ and $(E_2(v), w)$are measurable in y then $({}^{y}E(v), w)$ is also measurable in y where for all y, E_3 is the projection on the intersections of the ranges of E_1, and $'E_2$. To show This let $v_1,, v_2, \ldots$ be a dense subset of $\mathfrak{H}_0$. Then the range of' E_3 is the orthogonal complement of the sequence $(I - E_1)(v_1), (I - {}^{y}E_2)(v_1), (I - {}^{y}E_2)(v_2), (I - {}^{y}E_2)(v_2), \ldots$. Let ${}^{y}F_n$ be the projection of the orthogonal complement of the first n terms of this sequence.

Then $({}^{y}F_n(v), w)$ tends *to* $(E_3(v), w)$ as n tends to ∞ for all v and w in $\mathfrak{H}_0$. We need only show then that $({}^{y}F_n(v), w)$ is measurable for each n. This however can easily be established by induction on n.

As an immediate consequence of this lemma and the remarks preceding it we have [22]:

Theorem (1.2.37)[11]: Let $G_1, G_2, \mathfrak{G}$, L and M be as in Theorem (1.2.36). For each x and y in $\mathfrak{G}$ let J(L, M, x, y) be defined as in Theorem (1.2.36). Then J(L, M, x, y) depends only upon the double coset $D = D(x, y) = G_1 x y^{-1} G_2$ to which xy^{-1} belongs So that we may write J(L, M, D). Moreover whether or not $(x^{-1}G_1x) \cap (y^{-1}G_2y)$ is such that $\mathfrak{G}/((x^{-1}G_1x) \cap (y^{-1}G_2y))$ admits a finite invariant measure depends only on this double coset.

Let $\mathfrak{D}''$ be the set of all double cosets D such that (a) afinite invariant measure does exist, (b) J(L, M, D) > 0 (c) D is of measure zero. Let $D_1, D_2 \ldots$ be the double cosets of positive measure for which a finite invariant measure does exist. Then if $\mathfrak{D}''$ has Haar Measure different from zero we have $J(U^L, U^M) \ldots \infty$ and if $\mathfrak{D}''$ is of Haar measure zero then $J(U^L, U^M) = \sum_i J(L, M, D_i)$.

Let G_1 and G_2 be separable locally compact groups and let G_1 be Abelian. Let there be Given a homomorphism of G_2 into the group of automorphisms of G_1 and let us denote the map of $x\ e\ G_1$ under the automorphism associated with $y \in G_2\ by\ y[x]$. We assume that $x, y \to y[x]$ is continuous in both variables. Finally let $\mathfrak{G}$ be the set of all pairs x, y with $x \in G_1, y \in G_2$ and let $(x_1, y_1)(x_2, y_2) = (x_1 y_1[x_2], y_1 y_2)$. It may be verified without difficulty that $\mathfrak{G}$under this operation and with the Cartesian product topology is a locally compact separable topological group. Following Malcev we call $\mathfrak{G}$ the semi direct product of $G_1$and G_2 with respect to the given homomorphism. In [28] we have applied the principal Theorem to give an analysis of the irreducible representations of Such semi direct products. The discussion given there turns out to have been rather too concise and since it was further obscured by several confusing typo graphical errors it seems well to give a fuller version here. We shall proceed somewhat differently this time and make use of Theorem(1.2.34) .

We remark first of all that the set of all x, e where $x \in G_1$ and e is the identity is a closed normal subgroup of $\mathfrak{G}$ and is isomorphic in a natural manner to G_1. Similarly the set of all e, y for $y \in G_2$ is a closed subgroup of $\mathfrak{G}$ isomorphic in a natural manner to G_2. We shall identify G_1 and G_2 with the corresponding subgroups. Since $(x, e)(e, y) = (x, y)$ for all $x \in G_1$ and $y \in G_2$ it follows at once that The representation $x, y \to U_{x,y}\ of\ \mathfrak{G}$. is determined by its restrictions to G_1 and G_2. Indeed if $V(x \to V_x)$ and $W(y \to W_y)$ denote these restrictions then$U_{x,y} = V_x W_y$ Conversely if V and W are arbitrary representations of G_1 and G_2 respectively which act in the same Hilbert space then an easy calculation shows that $x, y \rightarrow V_x W_y$ defines a representation of $\mathfrak{G}$ if and only if $V_x W_y W_{y-1} = V_{y[x]}$ for all $x, y \in \mathfrak{G}$. Now by the Stone-Neumark-Ambrose-Godement spectral resolution Theorem [40], [27], [12], [20].V is determined by a projection valued measure $E \rightarrow P_E$.defined on theBoral subset of the character group $\hat{G}_1$ of G_2. It is readily verified that V and W satisfy the above identity if and only if P and W satisfy $W_Y P_E W_{y-1} = P_{[(B]y}$ for all $y \in G_2$ and all Borel sets E in $\hat{G}_1$. Here $[\hat{x}]_y$ is defined by the equation $(x, [\hat{x}]y) = (y[x], \hat{x})$. We leave it to the reader to verify that $\hat{x}, y \rightarrow [x]y$ has the to be expected elementary properties. In the terminology of [19] then P is a system of imprimitivity for W.

Consider the action of $G_2\ on\ \hat{G}_1$. If the projection valued measure P is concentrated in one of the orbits of $G_1\ under\ G_2$ let be any member of this orbit and let G_{x_0} be the subgroup of all $y \in G_2$for which $[\hat{x}_0]_y = \hat{x}_0$.

Then $y \rightarrow [\hat{x}_0]_y$ sets up a one-to-one Borel set preserving correspondence between the points of the orbit and the points of the homogeneous space $G_2/G_{\hat{x}_0}$. In this way P becomes a system of imprimitivity for W based on the homogeneous space $G_2/G_{\hat{x}_0}$and we may apply Theorem 2 of [19] to conclude that W is of the form $G_2 U^L$ where L is a representation of G_{x_0}.

Under certain often verifiable conditions it may be shown that U cannot be irreducible unless P is indeed concentrated in a single orbit.

Specifically let us say that $\mathfrak{G}$ a regular semi direct product of $G_1$and G_2 if $\hat{G}_1$ contains a countable family $E_1, E_2, \ldots$ of Borel sets each a union of orbits such that every orbit in $\hat{G}_1$ is the intersection of the E_j which contain it. As indicated in28]] it is easy to show that whenever$\mathfrak{G}$ is a regular semi direct product of G_1 and G_2 then a necessary condition for the irreducibility of the representation associated with P and W Is that there exist an orbit $\mathfrak{D}\ of\ \hat{G}_1$ under G_2 such that $P_E = 0$ whenever $E \cap \mathfrak{D} = 0$. Combining these considerations with Theorem 2 of19]] and the remarks about reducibility in $\neq$ 6 of19]] we may conclude the truth of

Theorem (1.2.38)[11]: Let $\mathfrak{G}$ be a regular semi direct product of the separable locally compact groups G_1, and G_2. LetG_1 be Abelian and let $\hat{G}_1$ be its character group.
From each orbit $\mathfrak{D}\ of\ \hat{G}_1$ under the action of G_2 choose an element $\hat{x}_0$ and let G_0 denote the set of all $y \in G_2$ such that $[\hat{x}_0]_y = \hat{x}_0. Let\ U(x, y \rightarrow U_{x,y} = V_x W_y)$ be an arbitrary irreducible representation of $\mathfrak{G}$. Then the projection valued measure defined by V in $\hat{G}_1$ is concentrated in a single orbit$\mathfrak{D}$and W is the representation $U^L\ of\ G_2$ induced by an irreducible

Representation of G_o . Every pair consisting of an orbit $\mathfrak{D}$ and an irreducible representation L of G_o arises from an irreducible representation $U\ of\ \mathfrak{G}$ in this way.
Finally two irreducible representations of $\mathfrak{G}$ are unitary equivalent if and only if the corresponding orbits are identical and the corresponding representations of G_0 are unitary equivalent.
We may also describe the representations of $\mathfrak{G}$ as follows.
Theorem (1.2.39)[11]: Conserving the notation of Theorem(1.2.38) let $\mathfrak{D}$ be an arbitrary orbit in $\hat{G}_1$and let L be an arbitrary irreducible representation of G_0. Let $\mathfrak{G}$. be the set of all

x, y with $y \in G_0$. Let M be the representation of $\mathfrak{G}_o$ defined by the equation $M_{x,y}(x, \hat{x}_o)L_y$. Then the representation U^{M} of $\mathfrak{G}$ induced by the representation M of $\mathfrak{G}_0$ is irreducible and in the unitary equivalence class associated with $\mathfrak{D}$ and L.

Proof: Let us apply Theorem(1.2.34) to study the restrictions V and W of U^{M} to G_1 and G_2 respectively. We verify without difficulty that the $\mathfrak{G}_o, G_1$ double cosets are in a natural one to one correspondence with the right $G_1/\mathfrak{G}_o$ D cosets and that $\mathfrak{G}_o$ and G_1 are regularly related. Moreover the representation of G_1 associated with the right G_1/G_o, coset containing $y \in G_2$ is the one dimensional representation defined by the member $[\hat{x}_0]_y$ of $\hat{G}_1$.

Thus V is adirect integral of characters belonging to the orbit $\mathfrak{D}$ and it follows easily that P is concentrated in $\mathfrak{D}$. Proceeding now to W we observe that there is only one $\mathfrak{G}_o$: U_2 double coset. Since $\mathfrak{G}_o \cap U_2 = G_0$ it follows now from Theorem(1.2.34) (actually from Theorem(1.2.18) that W is simply M.

All statements of the Theorem now follow from the preceding discussion.

Corollary (1.2.40)[11]:. If G_2 is Abelian then every irreducible representation of the regular semi direct product $\mathfrak{G}$is monomial; that is, is of the form U^{M} where M is a one dimensional representation of a subgroup of $\mathfrak{G}$.

Example (1.2.41)[11]: Let G_1, be the additive group of all real numbers and let G_2 be The multiplicative group of all positive real numbers. Let $y[x] = x \circ y$ where o denotes ordinary real number multiplication. Then $\mathfrak{G}$ is the so called "ax + b group" of linear transformations of the line.

$\hat{G}_1$ is again the additive group of the real numbers and there are just three orbits:$\mathfrak{D}_1$the set of all negatiye numbers $\mathfrak{D}_2$the origin, and $\mathfrak{D}_3$the set of all positive numbers. $G_{\circ_1}$and G$G_{\circ_3}$ consist of the identity alone; $G_{\circ_2}$is the, whole of G_2.

Applying Theorems (1.2.38)and(1.2.39) we see that in addition to the obvious one dimensional representations there are just two other irreducible representations and that each is infinite dimensional. Let L_1 be the one dimensional representation of G_1 associated with any member of $\mathfrak{D}_1$and let L_2 be similarly defined with respect to$\mathfrak{D}_3$. Then the two infinite dimensional representations of $\mathfrak{G}$ are the monomial representations U^{L_1} and U^{L_2}.

Example(1.2.42)[11]: Let G_1 be the additive group of all complex numbers and letG_2 be the multiplicative group of complex numbers of modulus one. Let $y[x] = x \circ y$ where o denotes ordinary multiplication of complex numbers. Then $\mathfrak{G}$ is group of all Euclidean motions of the plane. $\hat{G}_1$ is again the additive group of complex numbers and G_2 acts on $\hat{G}_1$ just as it does on G_1. Thus the orbits are the circles with center at the origin. Let $\mathfrak{D}_r$ be the orbit of radius r. If $r > 0$ then G_{o_r} is theidentity. If $r = 0$ then $G_{o_r} = G_2$.

Applying Theorems (1.2.38) and (1.2.39) we see that in addition to the obvious one dimensional representations there is a continuum of infinite dimensional irreducible representations one for each r > 0.

These exhaust the irreducible representations of $\mathfrak{G}$. The irreducible representation associated with $r > 0\ is\ U^{L_r}$ where L is a one dimensional representation of G_1 associated with a member of $\hat{G}_1$ of absolute value r.

Example (1.2.43)[11]: Let G_1 be the additive group of the plane and let G_2 be the multiplicative group of all two by two real matrices of the form $\begin{pmatrix} a & b \\ o & 1\backslash a \end{pmatrix}$where $a > 0$.

$If\ \hat{x} = \hat{x}_1, x_2$ and $y = \begin{pmatrix} a & b \\ o & 1\backslash a \end{pmatrix}$ $let\ [\hat{x}]\, y = \hat{x}_1 a + \hat{x}_2 b, \hat{x}_2/a$. Then $y[x]$ is uniquely defined and $\mathfrak{E}$ is a subgroup of the group of area preserving homeomorphisms of the plane. There are five orbits of $\hat{G}_1$ under G_2 ; $\mathfrak{D}_1$, is the positive real axis, $\mathfrak{D}_2$ is the negative real axis, $\mathfrak{D}_3$ is the origin alone, $\mathfrak{D}_4$ is the upper half plane and $\mathfrak{D}_5$ is the lower half plane. G_{o_1} and G_{o_2} are the subgroup of G_2 defined by setting $a = 1$. G_{o_3} is the whole of $G_2$and G_{o_4} and G_{o_5} are the identity. Noting that G_2 is isomorphic to the group of Example (1.2.41) we conclude easily that$\mathfrak{G}$ has a one parameter family of one dimensional representations and two infinite dimensional irreducible representations associated with the orbit $\mathfrak{D}_3$. Each of the orbits $\mathfrak{D}_1$, and $\mathfrak{D}_2$ has associated with it a one parameter family of irreducible infinite dimensional representations. $\mathfrak{D}_4$ and $\mathfrak{D}_5$ are associated with exactly one infinite dimensional irreducible representation each.

Example (1.2.44)[11]: Let G_1 be as in Example (1.2.43) and let G_2 be the additive group of integers. If y is an integer and x a complex number let y[x] be the product $z^y x$ where z is some fixed complex number of modulus one no power of which is one. In this case there are continuum many orbits on each circle with center at the origin in $\hat{G}_2$. On the other hand the only invariant Borel sets are essentially unions of circles. Thus $\mathfrak{G}$ is not regular and Theorems(1.2.38) and (1.2.39) do not apply. We shall study elsewhere the pathology presented by the representations of This group.

A few remarks are in order concerning the connection between Theorems (1.2.38) and (1.2.39). For finite groups results somewhat more general than Theorems(1.2.38) and(1.2.39) are classical. See for example Seitz [39], Shoda and the earlier work of Frobenius and Schur.

For infinite dimensional representations of non finite locally compact groups the only work we know of deals with particular groups.

Wigner in [8] shows that the study of the representations of the inhomogeneous Lorentz group may be reduced to the study of the representations of the homogeneous Lorentz group and certain of its subgroups.

In doing so he essentially shows Theorem (1.2.38) and /or(1.2.39) for the special case in which $G_1$is a vector group, G_2 is the homogeneous Lorentz group and y[x] is the result of transforming the point $x\ in\ G_1$ by the Lorentz transformation y. He also discusses Example (1.2.43) above. The representations of Example (1.2.41) above have been determined by Gelfand and Neumark [15] They assert that their method can be used to determine the representations of any solvable Lie group but do not formulate a general Theorem.

Actually the analysis of the group of Example (1.2.41) is considerably simpler than that of the general regular semi direct product because of the fact that in this case every G_o is either the identity or the whole of G_2. Whenever this happens the representations may be deduced from Theorem 2 of [27] and there is no need for the more general Theorem 2 of [28]. It should also be pointed out that there exist solvable Lie groups which are irregular semi direct products. For these one can show that there are many more irreducible representations than those described in Theorems(1.2.38) and(1.2.39) . Moreover their nature is such that one can well despair of ever obtaining a classification for them as complete and satisfying as that available for regular semi direct products in general and the group of Example (1.2.41) in particular. In Wigner's section the "factor representations" as well as the irreducible representations are discussed. Here by a factor representation is meant a representation U

such that the weakly closed ring generated by the operators U_x has only multiples of the identity in its center and hence is a "factor" in the sense of the definition of Murray and von Neumann. It has been pointed out to us by I. Kaplansky that our discussion of irreducible representations of regular semi direct products applies almost without change to factor representations.
One still finds that the projection valued measure must be confined to a single orbit and that V is of the form U^{L}. The only change lies in the fact that L need only be a factor representation itself and need not be irreducible. In particular it is easy to show the following
Theorem (1.2.45)[11]: Let 𝔊? be a regular semi direct product of G_1 and G_2. Suppose that G_2 and its closed subgroups have no factor representations except those of type I. Then 𝔊 has no factor representations that are not of type I. We applying Theorem (1.2.35) to compute the Kronecker products of the infinite dimensional irreducible representations in Examples (1.2.41) and (1.2.43). The computations themselves are straightforward and we shall content ourselves here with an enumeration of results.
Example (1.2.46)[11]: $U^{L_1} \otimes U^{L_2}$ is a direct integral over G_2 of replicas of U^{L_1}. $U^{L_1} \otimes U^{L_1}$ is a direct integral over G_2 of replicas of U^{L_2}. $U^{L_2} \otimes U^{L_2}$ is the same as $U^{L_1} \otimes U^{L_1}$. The measure is Haar measure in G_2. However this is really irrelevant. A direct integral of replicas of the same irreducible representation is always unitary equivalent to a direct sum of finite or countably many such replicas.
Example (1.2.47)[11]: Let us denote the irreducible representation associated with the orbit of radius r by W^r. Then $W^{r_1} \otimes W^{r_2}$ is a direct integral with respect To Haar measure in the reals mod 2π of the representations $W \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 c_0\, s^{\theta}}$. Alternately $W^{r_1} \otimes W^{r_2}$ is the integral over the interval $|r_1 - r_2| \leq r \leq r_1 + r_2,$ with respect to Lebesgue measure, of the direct sum of two replicas of W^r.

We shall indicate some of the connections between the theory of induced representations and the analysis given by Gelfand and Neumark in [16] of the representations of the group 𝔊 of all two by two complex matrices of determinant one. Following Gelfand and Neumark let us denote by K the set of all elements of 𝔊 of the form $\begin{pmatrix} a & b \\ o & 1\backslash a \end{pmatrix}$ where $a \neq 0$; by Z the set of all elements of K with a = 1 and by D the set of all elements in K with b = 0. Then K, Z and D are all closed subgroups of 𝔊 Moreover Z is a normal subgroup of K and every element of K is uniquely of the form zd where $z \in Z$ and d e D. Thus K is a semi direct product of the two Abelian groups Z and D. The automorphism of Z induced by $\begin{pmatrix} a & b \\ o & 1\backslash a \end{pmatrix}$ in D is $\begin{pmatrix} 1 & b \\ o & 1 \end{pmatrix} \to \begin{pmatrix} 1 & a^2 b \\ o & 1 \end{pmatrix}$. Let us denote the one dimensional representations of D by L, L_1,etc. and let us denote the one dimensional representations of Z by M, M_1etc.
Since D is isomorphic in a natural way to the quotient group K/Z each L defines a representation of K which we shall denote by L'. We shall be concerned with the induced representations $U^L, U^{L'}$ and U^M of 𝔊. The reader will have little difficulty in verifying that the representations U^{L_1} for variable L in $\widehat{D}$ constitute precisely what Gelfand and Neumark call the "principal series" of irreducible representations of 𝔊 and that U^{m} when M is the trivial representation of Z is what Gelfand and Neumark call the quasi regular representation

of $\mathfrak{G}$. Among the results of Gelfand and Neumark on the principal series and the quasi regular representation are the following.

I. Every member of the principal series is irreducible.

II. $U^{L'_1}$ and U_2 are unitary equivalent if and only if either $L_1 = L_2 \, or \, L_1 = \bar{L}_2$.

III. The quasi regular representation is a direct integral over the character group of D of the representations $U^{\mathrm{L}'}$.

IV. The regular representation of $\mathfrak{G}$ is a direct integral over the character group of D of $\mathfrak{G}$ fold repetitions of $U^{\mathrm{L}'}$.

We shall obtain I and III as consequences of our general theory and in addition certain results not obtained by Gelfand and Neumark. We hope to obtain II and IV as consequences of further general Theorems on induced representations, work upon which is now in progress.

A. The representations $U^{\mathrm{L}'}$ are not only irreducible but restricted to the subgroup K are irreducible representations of K.

Proof: An easy calculation shows that there are only two K: K double cosets in $\mathfrak{G}$. Since K itself has Haar measure zero there is effectively only one double coset.

Taking $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as the x of Theorem (1.2.45) we find that $x^{-1}Kx \cap K \, is \, D$ and that $\eta \to L'_x \, \eta x^{-1}$is the representation $\bar{L}$ of D. Thus $U^{L\prime}$ restricted to K is the representation $K^{U^{\bar{\mathrm{L}}}}$ of K induced by the representation $\bar{L} \, of \, D$. Applying Theorem(1.2.19) again we find that $K^{U^{\bar{\mathrm{L}}}}$ restricted to Z is the representation of Z induced by the one dimensional representation of the identity subgroup; that is the regular representation of Z. On the other hand there are in Z under D exactly two orbits; zero and everything else. The projection valued measure induced by the regular representation gives measure zero to the origin. Thus this measure is concentrated in a single orbit. Moreover the subgroup of D which leaves a point in this orbit fixed is the two element center C of $\mathfrak{G}$. Applying Theorem (1.2.18) a third time and observing that there is only one D: D double coset in K other than D itself and that D has measure zero we find that $K^{U^{\mathrm{L}}}$restricted to D is the representation of D induced by the restietion of $\bar{L} \, to \, C$. Thus $K^{U^{\bar{\mathrm{L}}}}$ is one of the two infinite dimensional irreducible representations of K associated with the non finite orbit in $\hat{Z}$.

B. (III above) If M is the identity representation of Z then U^{M} is a direct integral over $\widehat{D}$ of the representations $U^{L'}$ the measure being Haar measure in $\widehat{D}$.

Theorem (1.2.48)[11]: Let $\mathrm{G}_1 \subseteq \mathrm{G}_2$ be closed subgroups of the separable locally compact group$\mathfrak{G}$. Let L be a representation of G_1 and let $\mathrm{M} = \mathfrak{G}_2{}^{U^L}$. Then $\mathfrak{G}^{U^L}$ and $\mathfrak{G}^{U^M}$ are unitary equivalent representations of $\mathfrak{G}$.

When G_1 is the subgroup of $\mathfrak{G}$ which contains only the identity and L is the trivial one dimensional representation of G_1 then as is easily seen $\mathfrak{G}^{U^L}$is the regular representation of $\mathfrak{G}$. From this remark and Theorem (1.2.48) we deduce at once:

Proof: U^{M} is unitary equivalent to U^V where $V = K^{U^{\widehat{M}}}$ is the representation of K induced by the representation M of Z. Since M is the identity the space of K^{U^M} may be identified with $\mathfrak{L}^2(D)$. Thus M is the representation of K defined, via the natural homomorphism of K on D, by the regular representation of D. It follows from the theory of local compact Abelian groups however that the regular representation of D is simply a direct integral with respect to Haar measure in D of the irreducible representations L of D. The representation V of K is correspondingly the direct integral of the L'. Finally then by Theorem (1.2.48), $U^{M} = U^{V}$ is a direct integral of the $U^{L'}$ as stated.

We saw in the proof of A that K has only two infinite dimensional irreducible representations. Let us denote the one associated with the identity representation of C by W_1 and the other by W_2. Then we have

(C) If M is any irreducible representation of Z other than the identity then U^{M} is isomorphic to the direct sum of U^{W_1} and U^{W_2}. In particular U^{M} (for M not the identity) is independent of M..

Proof. As in B U^{M} is unitary equivalent to U^{V} where $V = K^{U^{M}}$. When M is not the identity it follows from Theorem (1.2.34) that $K^{U^{M}}$ restricted to Z is a direct integral of all one dimensional representation of Z which are distinct from the identity. Thus the projection valued measure associated with $K^{U^{M,Z}}$ is concentrated in the orbit of $\hat{Z}$ consisting of the complement of the origin.

On the other hand $K^{U^{M}}$ restricted to D is the regular representation of D and hence the representation of D induced by the regular representation of C.

Now the regular representation of C is of course simply the direct sum of its two irreducible representations. Thus $K^{U^{M}}$ is the direct sum of the two infinite dimensional irreducible representations W_1 and W_2 of K and the truth of our assertions follow.

D. If L is a one dimensional representation of D then $\mathfrak{G}^{U^{\mathrm{L}}}$ is unitary equivalent to U^{W_1} if L restricted to C is the identity. Otherwise $\mathfrak{G}^{U^{\mathrm{L}}}$ is unitary equivalent to U^{W_2}.

Proof. It follows from Theorem (1.2.48) that $\mathfrak{G}^{U^{\mathrm{L}}}$ is unitary equivalent to U^{V}

Where $V = K^{U^{\mathrm{L}}}. But\ K^{U^{\mathrm{L}}}$ was identified in the proof of A as being $W_1 or\ W_2$ according to whether or not L on C reduces to the identity.

E. The regular representation of $\mathfrak{G}$ is a direct sum of countably many replicas of U^{W_1} and countably many replicas of U^{W_2}.

Proof. By the Corollary to Theorem(1.2.27) the regular representation of $\mathfrak{G}$ is a direct integral with respect to Haar measure in $\widehat{D}$ of the U^{L}.

Thus we need only apply D above and remember that a direct integral of replicas of the same representations is equivalent to a discrete direct sum of the same replicas.

F. Let $U^{L_1'}$ and U^{L} be any two members of the principal series of irreducible Representations of $\mathfrak{G}$. If L_1 and L_2 are the same when restricted to C then the Kronecker product $U^{L_1} \otimes U^{L_2}$ is unitary equivalent to U^{W_1}. If L_1 and L_2 are distinct on C then $U^{L_1} \otimes U^{L_2}$ is unitary equivalent to U^{W_2}.

Proof. As we have already noted there are only two K:K double cosets and one has measure zero. Applying Theorem(1.2.19) we find at once that $U^{L} \otimes U^{L_2}$ is unitary equivalent to $U^{L_1\bar{L}_2}$. We now need only apply D.

In order to complete the considerations in a satisfactory manner we should want to know more about the two representations U^{W_1} and U^{W_2}. If we knew how to decompose them as direct integrals of members of the principal series we would have a complete analysis of the Kronecker product of any two members of this series as well as of the induced representations associated with the various subgroups of $\mathfrak{G}$ that we have considered. E combined with result IV of Gelfand and Neumark suggest that thedirect sum of $U^{W_1}$and U^{W_2} is either the regular representation itself or a representation which differs from the regular representation only in the multiplicity of occurrence of its components Just what is the case we do not know at this writing.

We remark that the Lorentz group is, as is well known, the quotient $\mathfrak{G}/C$. Because of this it is easy to derive results about it from results about $\mathfrak{G}$. In particular one can show that any two members of the principal series of irreducible representations of the Lorentz group have the same Kronecker product as any other two members.

Let $\mathfrak{G}$ be the group of all n by n complex matrices of determinant one and let U be the subgroup of all unitary matrices in $\mathfrak{G}$. Gelfand and Neumark in [18] and [19] have discussed certain relationships between representations of $\mathfrak{G}$ and representations of U. We shall show here that their principal results are corollaries of our Theorems (1.2.18) and (1.2.23) .

Let K be the subgroup of $\mathfrak{G}$ consisting of all matrices which vanish below the main diagonal and let Z be the subgroup of K consisting of all matrices in K which are one on the main diagonal. Then Z is a normal subgroup of K whose quotient K/Z is isomorphic in a natural manner to the group D of all diagonal matrices of $\mathfrak{G}$. Every one dimensional representation L of D thus defines a one dimensional Representation $L'\, of\, K$. We consider the induced representations $U^{L'}\, of\, \mathfrak{G}$. These are, just as in the two by two case, the members of what Gelfand and Neumark call the principal series and have been shown by them in [17] to be irreducible. Three of the four principal results of [12] slightly reformulated are:

A necessary and sufficient condition that $U^{L'}$ restricted to $\mathfrak{U}$ contain the identity as a discrete direct summand is that L reduce to the identity on $\mathfrak{U} \cap D = \Gamma$. If $U^{L'}$ does contain the identity it contains it exactly once.

III. Let M be an irreducible representation of $\mathfrak{U}$. Then M is contained in $U^{L'}$ restricted to $\mathfrak{U}$ if and only if M restricted to Γ contains the restriction of L to Γ.

IV. Let M be an irreducible representation of $\mathfrak{U}$ which is contained in $U^{L'}$ restricted to $\mathfrak{U}$. Then it is contained exactly as many times as M restricted to r contains the restriction of L to Γ. Their result II gives an explicit formula for the generator of the subspace of $\mathfrak{H}(U^{L'})$ in Which $U^{L'}$ reduces to the identity on $\mathfrak{U}$. In order to obtain proofs of I, III, and IV by our methods we note first that they may be combined into the following single Theorem.

Theorem (1.2.49)[11]: Let M be an irreducible representation of $\mathfrak{U}$. Let L be a one dimensional representation of D. Then the number of times that $U^{L'}$ restricted to U contains M as a discrete direct summand is equal to the number of times that M restricted to Γ contains L restricted to Γ.

Proof: Observe that there is only one $K{:}\,\mathfrak{U}$ double coset. Indeed let X be any member of $\mathfrak{G}$ and let $\varphi_1, \varphi_2, \ldots\ldots \varphi_n$. be the vectors $1, 0, 0 \ldots\ 0;\ 0, 1, 0, \ldots\ 0, 0 \ldots;1$ Let $\psi_1, \psi_2, \ldots, \psi_n$be a set of orthonormal vectors such that for each $i = 1,2, \ldots\ n$; $\psi_1 \ldots \psi_i$ span the same space as $X^{-1}(\varphi_1) \ldots\ X^{-1}(\varphi_i)$ and such that the unique unitary matrix Y such that $Y(\varphi_i) = \psi_i$ has determinant one. Then $XY(\varphi_i)X(\psi_i) = X\big(c_1X^{-1}(\varphi_1) + \cdots c_iX^{-1}(\varphi_i)\big) = c_1\varphi_1 \ldots . c_i\varphi_i$. Thus $XY \in K$ and since $Y \in \mathfrak{U}, X \in K\mathfrak{U}$. Applying Theorem (1.2.18) we conclude at once that $U^{L'}$ restricted to $\mathfrak{U}$ is the Representation of U induced by the representation L' restricted to $K \cap \mathfrak{U}$.

But $K \cap \mathfrak{U} = \Gamma$ and $\Gamma \subseteq D$. Thus $U^{L'}$ restricted to $\mathfrak{U}$ is the representation of $\mathfrak{U}$ induced by L restricted to Γ .The truth of the Theorem is now an immediate consequence of Theorem(1.2.23) .

Corollary (1.2.50)[260]: For each $x \in \mathfrak{G}$ the vectors $f^{r-1}(x)\ for\ f^{r-1} \in C_{L_{r-2}}^{r-1}$ form a dense linear sub- space of $\mathfrak{H}(L_{r-2})$.

Proof: Note first that if $f^{r-1} \in C_{L_{r-2}}^{r-1}$ and f_s^{r-2} is defined by the equation $f_s^{r-2}(x) = f^{r-2}(xs)$ for all x and s in $\mathfrak{G}$ then $(f^{r-1})_s(x) = (f_s)^{r-1}(x)$ so that for all f^{r-2} and s, $(f^{r-1})_s \in C_{L_{r-2}}^{r-1}$. Thus the set of vectors $f^{r-1}(x)$ for $f^{r-1} \in C_{L_{r-2}}^{r-1}$ and x fixed is independent of x. Let $\mathfrak{H}_1$ be the orthogonal complement of this set of vectors. Then if $v \in \mathfrak{H}_1$ we have $(f^{r-1}(x), v) = 0$ for all f^{r-1} and all x . Thus $(f^{r-1}(\xi x), v)) = (f^{r-1}(x), (L_{r-2})_{\xi^{-1}}(v))$ is zero for all f^{r-1} and x and all $\xi \in G$.

Hence $\mathfrak{H}_1$is invariant under the representation L_{r-2}. Let L'_{r-2} be the component of L in $\mathfrak{H}_1$. Suppose that there exists a non zero member f^{r-1} of $C_{L_{r-2}}^{r-1}$. Then $f^{r-1} \in C_{L_{r-2}}^{r-1}$ and we have a contradiction since the values of $f^{r-1}$are all in $\mathfrak{H}_1$. Thus in order to show that $\mathfrak{H}_1 = 0$ and complete the proof of the lemma we need only show that when $\mathfrak{H}_1 \neq 0$ there exists a non zero member f^{r-2} of $C_{L_{r-2}}^{r-1}$. But if none existed then $\int((L'_{r-2})_{\xi^{-1}}(f^{r-2}(\xi x)), v)\ dv(\xi)$ would be zero for all x, all v in $\mathfrak{H}(L_{r-2})$ and all f^{r-2} in C_L This is readily seen to be impossible.

Corollary (1.2.51)[260]: Let L_{r-2} and M_{r-2} be representations of the closed subgroups G_1 and G_2 of the separable locally compact groups $\mathfrak{G}_r$ and $\mathfrak{G}_{r+1}$ respectively. Then the representations $\mathfrak{G}_r \times \mathfrak{G}_{r+1}{}^{U^{L_{r-2} \times M_{r-2}}}$ and ${}_{\mathfrak{G}_r}U^{L_{r-2}} \times {}_{\mathfrak{G}_{r+1}} U^{M_{r-2}}$ of $\mathfrak{G}_r \times \mathfrak{G}_{r+1}$ are unitary equivalent.

Proof: Let T be a member of $\mathfrak{H}(U^{L_{r-2}} \times U^{M_{r-2}})$ [that is an operator from $\overline{\mathfrak{H}(\mu^2 \mathrm{U}^{M_{r-2}})}$ to $\mathfrak{H}(\mu_1 U^{L_{r-2}})$] whose range is finite dimensional.Then there exist $f_1, f_2, \dots f_n \in H({}^{\mu_1}U^{L_{r-2}})$ and $g_1, g_2 \dots . g_n \in\ , \mathfrak{H}({}^{\mu_2}U^{M_{r-2}})$ such that for each $g \in\ , \mathfrak{H}({}^{\mu_2}U^{M_{r-2}})$ we have $T(g^*) = (g_1, g)f_1 + \cdots + (g_n, g)f_n$. For each $x, x+\epsilon \in \mathfrak{G}_r \times \mathfrak{G}_{r+1}$ we may define an operator $A_T(x, x+\epsilon)$ from $\overline{\mathfrak{H}(M_{r-2})}$ to $\mathfrak{H}(L_{r-2})$ as follows. $(A_T(x, x+\epsilon)((x+3\epsilon)^*) = f_1(x)(g_1(x+\epsilon), x+3\epsilon) + \cdots + f_n(x)(g_n(x+\epsilon), x+3\epsilon)$. We note at once that $A_T(\xi x, \eta(x+\epsilon)) = (L_{r-2})_\xi A_T(x, x+\epsilon)(M_{r-2})_\eta^*$ for all $x, x+\epsilon \in \mathfrak{G}_r \times \mathfrak{G}_{r+1}$ and all $\xi, n \in G_1 \times G_2$. Moreover $|||A_T(x, x+\epsilon)|||^2 = \sum_{ij} \left(f_i(x), f_j(x)\right)(g_i(x+\epsilon), g_i(x+\epsilon))$ and

$$
\begin{aligned}
|||T|||^2 &= \sum_{ij} (f_i : f_j)(g_j : g_i) \\
&= \sum_{ij} \left(\int (f_i(x), f_j(x)) d\mu_1(x+2\epsilon)\right)\left(\int (g_j(x+\epsilon), g_i(x+\epsilon) d\mu_2(x+2\epsilon)\right) \\
&= \int \left(\sum_{i,j} (f_i(x), f_j(x)\right)(g_j(x+\epsilon), g_i(x+\epsilon)) d(\mu_1 \times \mu_2)(x+2\epsilon) \\
&= \int |||A_T(x, x+\epsilon)|||^2\, d(\mu_1 \times \mu_2)(x+2\epsilon).
\end{aligned}
$$

Corollary (1.2.52)[260]: Let U and $U+\epsilon$ be representations of the separable locally compact group $\mathfrak{G}$. Then J(U, $U+\epsilon$) (D(U)) $=$ I(^{0}U,^{0} $U+\epsilon$) and this number is equal to the number of times that $U\otimes\overline{(U+\epsilon)}$ contains the identity representation as a discrete direct summand; that is the dimension of the subspace of $\mathfrak{H}(U)$ in which all U_{x^2}.act as the identity.

Proof. If $U_{x^2}T = T(U_{x^2}+\epsilon)$ then $U_{x^2}T\left(U_{x^2}+\epsilon\right)^{-1} = T$ which may be written $U_{x^2}T^*_{x^2} = T$ or $\left(U\otimes(\widetilde{U+\epsilon})\right)_{x^2}(T) = T$. Since all steps are reversible the equality of J(U, $U+\epsilon$) to the dimension of the identity component of $U\otimes(\widetilde{U+\epsilon})$ is established. We now show the equality of J(U, $U+\epsilon$) and $I(^0U,^0U+\epsilon)$. Let T be any strong intertwining operator for U and $U+\epsilon$. Let M_1 be the orthogonal complement of the null space of T and let M_1 be the closure of the range of T. Since T is an intertwining operator it follows that $M_1$and M_2 are invariant under U and $U+\epsilon$ respectively. Let $A(v) = \left(T^*(T(v)^*)\right)^*$. Then A is a self adjoint operator in $\mathfrak{H}(U+\epsilon)$ which commutes with all $U_{x^2}+\epsilon$ and is completely continuous. Because of the latter property it has a pure point spectrum and each non zero value occurs only a finite number of times. It follows that M_2 is a direct sum of finite dimensional invariant subspaces and a similar argument shows that the same is true of M_1. Thus $M_2 \subseteq (\mathfrak{H}(U+\epsilon))_f$ and $M_2 \subseteq$(&(U))f . Hence every strong intertwining operator carries $(\mathfrak{H}(U+\epsilon))_f$ into $(\mathfrak{H}(U))_f$fand is zero on the orthogonal complement of $(\mathfrak{H}(U+\epsilon))_f$ it follows at once that $J(^0U,^0U+\epsilon) = J(U;\, U+\epsilon)$. Finally it is evident that both $I(^0U,^0U+\epsilon)$ and $J(^0U,^0U+\epsilon)$are equal to $\sum_{W^{n}W^{m}W}$ where the sum is over all finite dimensional irreducible representations of $\mathfrak{G}$ which appear as components of either $^0U\ or\ ^0U+\epsilon$, and where n_W $(resp. m_W)$ is the multiplicity of occurrence of W in ^0U (resp. $^0U+\epsilon$).

Chapter 2
Kadec Norms and Borel Sets

We characterize the existence of Kadec type renormings in the spirit of the new results for LUR spaces by Molto, Orihuela and Troyanski. It is also shown that a non-coincidence of norm-Borel and weak-Borel sets in a function space does not simply that the duality map is non-Borel.

Section (2.1): Borel Sets in a Banach Space

$(X, \|\cdot\|)$ will denote a Banach space, X^* its dual, w and w^* the weak and weak* topologies respectively, $B_X(resp.\ B_{X^*})$ denotes the unit ball of X $(resp.\ X^*)$. S_X will be the unit sphere of X. We shall also consider topologies on X of convergence on some subsets of the dual space. A subset of B_{X^*} is said to be norming (resp. quasi-norming) if its $w^* - closed$ convex envelope is B_{X^*} (resp. if the envelope contains an open ball centered at the origin).

A norm $\|\cdot\|$ on X is said to have the Kadec property when the weak and norm topologies coincide on the unit sphere. A norm is said to be locallyuniformly rotund (LUR) if for every sequence (x_n)in the unit sphere and for every point x in the unit sphere such that $\lim_n \|x_n + x\| = 2$ the sequence (x_n) converges to x in norm. LUR norms have the Kadec property. For the proof of this fact other properties of Banach spaces having an equivalent LUR norm see [47]. There exist Banach space shaving a Kadec norm and admitting no equivalent LUR norm [54].

Edgar [48] showd that in a Banach space which admits an equivalent Kadec norm the Borel $\sigma -$algebras generated by the weak and norm topologies coincide. He also noted that an analogous result also holds when the Kadec property holds for the weak* topology.
Schachermayer [49] showd that a Banach space X that has an equivalent Kadec norm is a Borel set in (X^{**}, w^*). Talagrand [69] showed that the previous two results are not true for general Banach spaces, but he showd [68] that for subspaces of weakly compactly generated spaces the Borel sets for the topology of point wise convergence on a quasi-norming subset of the dual space and the norm Borel sets are the same.

Jayne, Namioka and Rogers [58] introduced the nation of a countable cover by sets of small local d-diameter (SLD) (see Definition (2.1.3)) for a topological space with respect to some metric d and they noted that if a Banach space X has an equivalent Kadec norm then (X, w) has SLD with respect to the norm, which implies the coincidence of the Borel sets for the norm and weak topologies. In fact, property SLD implies the coincidence of the Borel sets for the original topology and the metric in a wider topological context .Oncina [65] has made adeep study of property SLD showing that a Banach space with SLD for the weak topology with respect to the norm is Borel set in its bidual . Another approach to the coineidence of the Borel set and related properties has been given by Hansel [53] using the nation of descriptive topological space. In the context of a Banach space endowed with its weak topology, Hansell's nation of descriptive space is equivalent to property SLD, as pointed out by Molto, Orihuela, Troyanski and Valdivia [53].

Molto, Orihuela, and Troyanski[62] have characterized the Banach spaces which admit an equivalent LUR norm as those spaces X such that (X, w) satisfies a special case of norm SLD: X has an equivalent LUR norm if and only if (X, w) satisfies Definition (2.1.3) below and the weak neighborhood there is s slice (the intersection with an open half space). See also the comments after Theorem (2.1.14).

We show that all the above mentioned positive results on coincidence of Borel σ-algerbras and the Borel nature of a Banach space in its bidual stem from a common topological principle which can be used to characterize the existence of Kadec type norms in a Banach space.

We introduce a useful condition Definition (2.1.1) for a couple of topologies that gives a natural approach to the study of Borel sets Proposition (2.1.5). When one of the topologies is given by a metric, our property is equivalent to property SLD Definition (2.1.3), Proposition (2.1.4).

We use the framework of topological vector spaces to study the relation between property SLD and the existence of Kadec type equivalent norms. We show that if X is a Banach space such that (X, w) has SLD then the weak and norm topologies coincide on the level sets of some positive homogeneous function Theorem (2.1.13). We also characterize the existence of an equivalent Kadec Theorem (2.1.14) in the spirit of the recents results on LUR norms by Molto, Orihuela and Troyanski [61]

We apply the previous results to WCD Banach spaces taking advantage of the existence of a LUR norm to build Kadec norms for topologies weaker than the topology Theorem (2.1.15) and to show the coincidence of Borel sets improving a result by Talagrand. As an application to nonmetric topologies we finish by showing that if K is a Radon-Nikodym compact set then $C(K)$ has an equivalent norm such that the weak and pointwise topologies coincide on the unit sphere (Theorem (2.1.18).

Parts of the results have been announced in [66]

Actually the idea is implicit in [68]. We recall that a network for some topology is a family of sets not necessarily opens such that every open set can be written as a union of sets in the family.

Definition (2.1.1)[43]: Let X be a set, and $\tau_1$and τ_2 two topologies on X. A subset $A \subset X$ is said to have property $P(\tau_1,\ \tau_2)$ if there exists a sequence (A_n) of subsets of X such that the family $(A_n \cap U)$ where $n \in \mathbb{N}$ and $U \in \tau_2$is a network for τ_1, that is, for every $\mathcal{X} \in A$ and every $V \in \tau_1$ with $\mathcal{X} \in V$ there exist $n \in \mathbb{N}$ and $U \in \tau_2$ that $\mathcal{X} \in A_n \cap U \subset V$.

Evidently, if $\tau_1 \subset \tau_2$ then X has $P(\tau_1,\ \tau_2)$, but this case is not interesting. The relevant case happens when $\tau_2 \subset \tau_1$, for instance, in applications to Banach spaces τ_1 and τ_2 will be the norm and the weak topology respectively. If τ_1 has a countable basis (V_n) then X has $P(\tau_1,\ \tau_2)$ for any τ_2, because we can take $A_n = V_n$.this happens in particular when $(X,\ \tau_1)$ is metrizable and separable. In fact, we shall use the property introduced in Definition (2.1.1) to extend results valid for separable spaces to no separable spaces.

If we take the sequence $(A_n \cap A)$ we can always suppose that $A_n \subset A$. That means that property $P(\tau_1,\ \tau_2)$ only depends on A equipped with the relative topologies.

To check $P(\tau_1,\ \tau_2)$ for a given A it is enough to verify the above set inclusion for all the V's belonging to a sub-basis of τ_1, because then A will have $P(\tau_1,\ \tau_2)$ with the countable family of the finite intersections of sets of the sequence (A_n).

The following proposition contains some other elementary consequences of Definition (2.1.1).

Proposition (2.1.2)[43]: Let X be v a set, $\tau_1,\ \tau_2\ \ and\ \tau_3$topologies on X, and A a subset of X. Then:

(i) If A has $P(\tau_1,\ \tau_2)$ and $B \subset A$ then B has $P(\tau_1,\ \tau_2)$.

(ii) If A has $P(\tau_1,\ \tau_2)$ and $P(\tau_2,\ \tau_3)$ then A has $P(\tau_1,\ \tau_3)$.

(iii) If every point of A has a τ_1 −basis of neighbourhoods which is made up of τ_2-closed sets then the sequence (A_n) in Definition(2.1.1) can be taken to consist of τ_2- closed sets.

(iv) If every set A_n of Definition (2.1.1)is τ_2-Borel then for every $V \in \tau_1$ such that $A \subset V$, there is a τ_2 −Borel set B satisfying $A \subset B \subset V$. In particular, if A is τ_1 −open , or more generally, if A is a G_δ −set for the τ_1 −topology, then A is τ_2 −Borel.

Proof: (i) Use the same sequence(A_n).

(ii) If (B_m) is a sequencefor $P(\tau_2,\tau_3)$ then it is easy to check that $(A_n \cap B_m)$ satisfies the condition of Definition(2.1.1) for $P(\tau_1,\tau_3)$.

(iii) Fix $\mathcal{X} \in A$. Take $V \in \tau$ with $\mathcal{X} \in V$. Take $V_0 \in \tau_1$ such that $\mathcal{X} \in V_0$ and $\bar{V}_0^{\tau_2} \subset V$. There exist A_n and $U \in \tau_2$ such that $\mathcal{X} \in A_n \cap U \subset V_0$. Thus

$$\mathcal{X} \in \bar{A}_{\tau_1}^{\tau_2} \cap U \subset \overline{A_n \cap U}^{\tau_2} \subset V.$$

(iv) For every $\mathcal{X} \in A$ there exist $n_{\mathcal{X}} \in \mathbb{N}$ and $U_{\mathcal{X}} \in \tau_2$ such that $\mathcal{X} \in A_{n_{\mathcal{X}}} \cap U_{\mathcal{X}} \subset V$. Now we have

$$A = \bigcup_{\mathcal{X}\in A} \{\mathcal{X}\} \subset \bigcup_{\mathcal{X}\in A} A_{n_{\mathcal{X}}} \cap U_{\mathcal{X}} = \bigcup_{n=1}^{\infty} \left(A_n \cap \bigcup_{n_{\mathcal{X}}=n} U_{\mathcal{X}} \right) = B \subset V$$

Where B is clearly in Borel (X,τ_2).

If $A = \bigcap_{n=1}^{\infty} V_n$ where $V_n \in \tau_1$we can take τ_2 −Borel sets (B_n)such that $A \subset B_n \subset V_n$. Then $A = \bigcap_{n=1}^{\infty} B_n$.

A particularly interesting case occurs when $\tau_1$is metrizable. In this case the property introduced in Definition(2.1.1) agrees with the following one given by Jayne, Namioka and Rogers in [58], which is a special case of their $\sigma -$ fragmentability.

Definition (2.1.3)[43]: Let (X,τ)be topological space and let d be a metric on X.T hen X has a countable cover by sets of small local diameter (SLD) if for every $\varepsilon > 0$ there exists a decomposition

$$X = \bigcap_{n=1}^{\infty} X_n^e$$

such that for each $n \in \mathbb{N}$ every point of X_n^e has a relative $\tau -$ neighbourhoodof diameter less than ε.

A Banach space X is said to have countable Szlenk index if for every $\varepsilon > 0$, there is a decreasing transfinite countable sequence (C_α) of
subsets such that $B_X = U_\alpha(C_\alpha \setminus C_{\alpha+1})$ and every point of $C_\alpha \setminus C_{\alpha+1}$ has a relative weak neighbourhood in C_α of diameter less than ε, These spaces have been considered by Lancien [61] Clearly, if X has countable Szlenk index, then (X,w) has $\|\cdot\|$- SLD. However, a separable Banach space X without the Point of Continuity Property does not have countable Szlenk index but (X,w) has $\|\cdot\|$- SLD.

Proposition (2.1.4)[43]: Let (X,τ)be a topological space and d a metric on X. Then X has a countable cover by sets of small local diameter if and only if X has $P(d,\tau)$. Moreover, if the closed *d-balls* are τ-closed then the stets X_n^e in Definition (2.1.3) can be taken to be differences of τ-closed sets.

Proof: If X_n^e are the sets of Definition (2.1.3) it is easy to check that the sets(A_n) obtained by arranging $\left(X_n^{1/m}\right)_{n,m}$ into a sequence by a diagonal process satisfy the condition of Definition (2.1.1).

For the other implication, given $\varepsilon > 0$ just define

$$X_n^e = \{\mathcal{X} \in A_n : \exists U \in \tau, \mathcal{X} \in U, diam\,(A_n \cap U) < \varepsilon\}.$$

The "moreover" part is a consequence of Proposition (2.1.2) (iii).

The following result shows the good Borel behavior of a topological space (X,τ) *that has* $P(d,\tau)$ for some appropriate metric d. The statement (a) has already been noted by Jayne, Namioka and Rogers in [64] and [66], in terms of property SLD.

Proposition (2.1.5)[43]: Let (Y,τ)be a topological space and d a metric on Y stronger than τ and such that closed d-balls are τ-closed. Let X be a subset of Y having $P(d,\tau)$.

(a) Considering X with the inherited topologies we have

$$Borel(X,\tau) = Borel(X,d).$$

(b) If X is d-closed in Y then $X \in Borel(Y,\tau)$.

Proof: (a) Evidently every τ-Borel set is a d-Borel set. Conversely, if $V \subset X$ is a d-open set then it has $P(d,\tau)$. as closed d-balls are τ-closed we can apply Proposition (2.1.2) (iii), (iv) to conclude that V is τ-Borel.

(b) Since X is a G_δ-set in (Y, d), the result follows from Proposition (2.1.2) (ii), (iv).

The next corollary embraces the applications of property SLD to Banach spaces by Jayne, Namioka and Rogers [58], Oncina [65] and Hansell [59] (this last using the notion of descriptive space) that imshow preceding ones by Edgar [48] and Schachermayer [49] on Banach spaces admitting Kadec norms. We shall show later that Banach spaces having $P(\|\cdot\|,\tau)$ are not very different from Banach spaces that admit an equivalent Kadec norm (Theorem (2.1.13).

Corollary (2.1.6)[43]: let X be a Banach space and τ a vector topology weaker than the norm topology and such that $\bar{B}_X^\tau$ is bounded.

(a) If X has $P(\|\cdot\|,\tau), then\ Borel\ (X,\|\cdot\|) = Borel(X,\tau)$.

(b) If X has $P(\|\cdot\|,w), then\ X \in Borel\ (X^{**},w^*)$.

Proof: Note that $\bar{B}_X^\tau$ is the unit ball of an equivalent norm on X whose closed balls are τ-closed. Then apply Proposition(2.1.5) .

Let us remark that $\bar{B}_X^\tau$ is bounded, for instance, when τ is the topology of convergence on a norming or a quasi-norming subset of X^*.

We now give an application of Proposition (2.1.5) to descriptive topology.

Following Fremlin (see [59]), a completely regular topological space X is $\tilde{C}ech$-analytic if for every finite sequence s of positive integers there is a set A(s) open or closed in the $\tilde{C}ech$-Stone compactification of X such that

$$X=\bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}}\bigcap_{n=1}^{\infty}A(\sigma\setminus n)$$

where $\sigma \setminus n$ denotes the finite sequence made up from the first n terms of the sequence σ. The notion of $\tilde{C}ech$-analytic space has some interest of nonseparable and nonnetrizable topological spaces (e. g. a Banach space endowed with its weak topology), where the classical descriptive set theory is not applicable in general. See [59] and [53] for more information about $\tilde{C}ech$-analytic spaces and their applications to Banach spaces.

Corollary (2.1.7)[43]: Let (X,τ) be a topological space. Suppose that there is a set T such that X can be identified as a subspace of $\mathbb{R}^T$ with the pointwise topology which is made up of bounded functions and is complete for the metric d on X of uniform convergence on T. If X has $P(X,\tau)$, then X is a Borel subset of $\mathbb{R}^T$, in fact a pointwise $(F\cap G)_{\sigma\delta}$, as $\tilde{C}ech$-analytic.

Proof: We can assume that d is defined on $\mathbb{R}^T$ and it is stronger than the pointwise topology with pointwise d-closed balls. As complete for d, it is d-closed in $\mathbb{R}^T$ and we finish by applying the proofs of Propositions (2.1.2) and (2.1.5).

According to [59] a sufficient condition for (X,τ) to be $\tilde{C}ech$ -analytic is being homeomorphic to a Borel subset of some compact space. The reasoning above shows that $X\cap[-n,n]^T$ is Borel in $[-n,n]^T$, so it is Borel in $\overline{\mathbb{R}}^T$ where $\overline{\mathbb{R}}$ is the two-point compactification of $\mathbb{R}$ Now, as $X=\bigcup_{n=1}^{\infty}X\cap[-n,n]^T$ it is a Borel set in the compact $\overline{\mathbb{R}}^T$.

Hansell [59] shows that a descriptive topological space is always $\tilde{C}ech$-analytic, in particular, every Banach space X such that (X,w) has $\|\cdot\|$-SLD is $\tilde{C}ech$-analytic (see [63]). Corollary (2.1.7) contains more information about the structure of X in that particular case.

Under the hypothesis of Corollary (2.1.7), it is easy to show that every d-Borel subset of X is pointwise Borel in $\mathbb{R}^T$ and analogously $\tilde{C}ech$-analytic.

It is convenient for our purposes to give a more general definition of Kadec norms involving topologies different from the weak topology.

Definition (2.1.8)[43]: Let X be a Banach space and τ a vector topology weaker than the norm topology. An equivalent norm $\|\cdot\|$ is said to be τ-Kadec if the norm topology and τ coincide on the unit sphere of $\|\cdot\|$.

The next result appears in [44] .

Proposition (2.1.9)[43]: A τ-Kadec norm $\|\cdot\|$is τ-lower semi continuous, that is, its unit ball is always $\tau-$ closed.

Proof: Suppose that $\|\cdot\|$is not τ-lsc. Then there is a net $(\mathcal{X}_w)$ on the unit sphere S_X and a point $\mathcal{X}$ outside the unit ball B_X such that $\tau\text{-}\lim_w\mathcal{X}_w=\mathcal{X}$ take numbers $t_w>1$ such that $\|\mathcal{X}+t_w(\mathcal{X}_w-\mathcal{X})\|=\|\mathcal{X}\|$. Let $\mathcal{Y}_w=\mathcal{X}+t_w(\mathcal{X}_w-\mathcal{X})$. Note that $\{t_w\}$ is bounded because$inf_w\|\mathcal{X}_w-\mathcal{X}\|>0$.We deduce that $\tau\text{-}\lim_w\mathcal{Y}_w=\mathcal{X}$. Since $\|\mathcal{Y}_w\|=\|\mathcal{X}\|$ we should have $\lim_w\|\mathcal{Y}_w-\mathcal{X}\|=0$, but this is impossible because $\|\mathcal{Y}_w-\mathcal{X}\|\geq\|\mathcal{X}_w-\mathcal{X}\|$.

As mentioned, LUR norms provide examples of norms with the Kadec property. In fact, it is not difficult to show that a τ-lower semicontinuous LUR norm is τ-Kadec. At this point important to remark that if the unit ball of a Banach space is τ-closed for some vector topology τ, then the new unit ball after a renorming is not necessarily τ-closed. For example, there exists a dual Banach space that admits an equivalent LUR norm but no equivalent dual LUR norm (see the remark after Theorem (2.1.15).

Given two topologies τ_1 and τ_2 on X and a family Σ of subsets of X we shall say that Σ is good at $\mathcal{X} \in X$ $if\ for\ every\ V \in \tau_1\ with\ \mathcal{X} \in \tau_1$ there exist $S \in \Sigma\ and\ U \in \tau_2$ such that $\mathcal{X} \in S \cap U \subset V$. A good family means a family good at every point of X. It is easy to see that a family Σ covering X such that on every $S \in \Sigma$ the topologies $\tau_1\ and\ \tau_2$ coincide is good and property $P(\tau_1, \tau_2)$ is equivalent to the existence of a countable good family of " thick" sets from a good one made up of " thin" sets.

Lemma (2.1.10)[43]: Let X be a vector space, $\tau_2 \subset \tau_1$ vector topologies on X and Σ a family good at some $\mathcal{X} \in X$. Then the family

$$\{S + W : S \in \Sigma, 0 \in W \in \tau_1\}$$

is good at $\mathcal{X}$. Thus, if Σ and II are families of subsets of X such that for every $S \in \Sigma$ and every $W \in \tau_1$ with $0 \in W$ there exists $P \in$ II such that

$$S \subset P \subset S + \mathrm{W}$$

then II is good if and only if Σ is.

Proof: Given $V \in \tau_1\ \ with\ \mathcal{X} \in V$ we shall find $S \in \Sigma$, $0 \in \mathrm{W} \in \tau_1$ and $U \in \tau_2$such that

$$\mathcal{X} \in (S + W) \cap U \subset v,$$

as $0 + \mathcal{X} \in V$ we can take $W_1, V^1 \in \tau_1$ with $0 \in W_1, \mathcal{X} \in V^1$ and $W_1, +V^1 \subset V$. Since Σ is good at $\mathcal{X}$ there are $S \in \Sigma\ \ and\ U^1 \in \tau_1$ such that $\mathcal{X} \in S \cap' \subset V'. As\ \ 0 + \mathcal{X} \in U'$ we can find $W_2, U \in \tau_1$ with and $0 \in W_2, x \in U$and $W_2 + U \subset U'$. Now take$W = W_1 \cap (-W_2) \in \tau_1$. We show that U and W satisfy the above set inclusion. If $\mathcal{Y} \in (S + W) \cap U$ then there is $\mathcal{Z} \in S\ \ such\ that\ \mathcal{Y} - \mathcal{Z} \in W \subset -W_2\ \ so\ \mathcal{Z} = (\mathcal{Z} - \mathcal{Y}) + \mathcal{Y} \in U^1$ Thus

$$\mathcal{Z} \in S \cap U^1 \subset V^1.$$

Now as $\mathcal{Y} - \mathcal{Z} \in W \subset W_1$ we have $\mathcal{Y} = (\mathcal{Y} - \mathcal{Z}) + \ \mathcal{Z} \in V$.

The applications of Kadec type norms to the results developed are contained in the following lemma.

Lemma (2.1.11)[43]: Let $(X, \|\cdot\|)$ be a normed vector space, and $\tau_2 \subset \tau_1$ be vector topologies on X weaker than the norm topology. Suppose that there exists a positive homogeneous function F on X such that:

(a) $F(\mathcal{X}) \geq c\|\mathcal{X}\|$ for some $c > 0$.

(b) $\tau_{1\ and}\ \tau_2$ coincide on the set $S\{\mathcal{X} \in X : F(\mathcal{X}) = 1\}$.

Then X has $P(\tau_1, \tau_2)$. In particular, if X is a Banach space that admits an equivalent τ-Kadec norm for some weaker vector topology τ then X has $P(\|\cdot\|, \tau)$.

Proof: Consider the following families of sets: $\Sigma = \{S(t) : t \in [0, \infty)\}$ and the countable one $\Pi = \{A(\tau, s) : \tau, s \in \mathbb{Q},\ 0 \leq r \leq s\}$ where

$$S(t) = \{\mathcal{X} \in X : F(\mathcal{X}) = t\}, \quad A(r, s) = \{\mathcal{X} \in X : r \leq F(\mathcal{X}) \leq s\}.$$

If $W \in \tau_1$ is a neighbourhood of 0 then it contains some ball $B[0, \delta]$. It is easy to see that for δ small enough

$$S(t) \subset A(t - c\delta) \subset S(t) + W.$$

The result follows from Lemma (2.1.10).

Combining Proposition (2.1.2), Corollary (2.1.6) and the previous lemma we easily obtain the theorems of Edgar and Schachermayer. Note that a more direct proof of Edgar's theorem just needs a special case of Lemma (2.1.10) and the idea of point (iv) of Proposition (2.1.2). Schachermayer's theorem moreover needs Proposition (2.1.2).

Corollary (2.1.12)[43]: Let X be a Banach space that admits an equivalent Kadec norm. Then Borel $(X, \|\cdot\|)$ =Borel(X, w) and $X \in$Borel (X^{**}, w^{*}).

A partial similar result has been showd by Lancin [61]

Theorem (2.1.13)[43]: Let X be a Banach space and τ a vector topology coarser than the norm topology such that $\bar{B}_X'$ is bounded. Then the following are equivalent:

(i) X has $P(\|\cdot\|, \tau)(equivalent, (X, \tau) has\ \|\cdot\| - SLD)$.

(ii) There exists a nonnegative symmetric homogeneous τ-lower semi continuous function F on X with $\|\cdot\| \leq F \leq 3\|\cdot\|$ such that the norm topology and τ coincide on the set $S = \{\mathcal{X} \in X: F(\mathcal{X}) = 1\}$.

Proof: (ii) ⇒ (i). This is in fact Lemma(2.1.11) .

(i) ⇒ (ii). Assume that X is endowed with a τ-lower semicontinuous equivalent norm $\|\cdot\|, B(0, a)$ and $B[0, a]$ are the open and closed balls of center 0 and radius a. As usual $B_X = B[0, 1]$.

Suppose that X has $P(\|\cdot\|, \tau)$ with a sequence (A_n).We can suppose every A_n is star shaded with respect to 0 and norm open. To see that, we are going to modify the sequence in several steps.

STEP 1: take $A_n' = A_n \cap B_X$.

STEP 2: Take

$$A_n'' = \{t\mathcal{X}: 0 \leq t \leq 1, \mathcal{X} \in A_n'\}.$$

We now check that (A_n'') is good for the points of the unit sphere S_X. Let $\mathcal{X} \in S_X\ and\ \varepsilon > 0$. Applying Lemma(2.1.10) we can find $U \in \tau, n \in \mathbb{N}$ and $\delta > 0$ such that $\mathcal{X} \in A_n' \cap U$ and diam$\left((A_n' + B(0, \delta)) \cap U\right) < \varepsilon$. Now it is clear that

$$A_n'' \cap (U \setminus B[0, 1 - \delta]) \subset \left(A_n' + B(0, \delta)\right) \cap U.$$

Thus $U' = U \setminus B[0, 1 - \delta] \in \tau$ satisfies $\mathcal{X} \in A_n'' \cap U'$ and diam$(A_n'' \cap U') < \varepsilon$.

STEP 3: The family

$$\{rA_n'' + B(0, \delta): n \in \mathbb{N}, r \geq 0, \delta > 0, r, \delta \in \mathbb{Q}\}$$

is good for X by Lemma (2.1.10). Renumbering this family yields the desired (A_n).

Clearly the sets $\overline{A_n'}$ are star shaped with respect to 0. Let f_n be the Minkowski functional of $\overline{A_n'}$. $Since\ \overline{A_n'} = \{f_n \leq 1\}$ the function f_n is τ-lower semi continuous. Let $\|f_n\|$ be the supremum of $|f_n(\mathcal{X})|$ with $\mathcal{X} \in B_X$.The function F given by the formula

$$F(\mathcal{X}) = \|\mathcal{X}\| + \sum_{n=1}^{\infty} \frac{1}{2n} \cdot \frac{f_n(\mathcal{X})}{\|f_n\|} + \sum_{n=1}^{\infty} \frac{1}{2n} \cdot \frac{f_n(-\mathcal{X})}{\|f_n\|}$$

isτ-lower semicontinuous and symmetric.

Let $(\mathcal{X}_w) \subset S$ be a net τ-converging to some $\mathcal{X} \in S$. From the τ-lower semi continuity of $\|\cdot\|\ and\ f_n$ we have

$$\|\mathcal{X}\| \leq \lim_w inf \|\mathcal{X}_{w},\|$$
$$f_n(\mathcal{X}) \leq \lim_w \inf f_n\,(\mathcal{X}_w),$$
$$f_n(-\mathcal{X}) \leq \lim_w \inf f_n\,(-\mathcal{X}_w),$$

On the other hand, it is not difficult to see that

$$1 \geq \lim_w \inf \|\mathcal{X}_{w},\| + \sum_{n=1}^{\infty} \frac{1}{2^n \|f_n\|} \lim_w \inf f_n(\mathcal{X}_w) + \sum_{n=1}^{\infty} \frac{1}{2^n \|f_n\|} \lim_w \inf f_n(-\mathcal{X}_w)$$

Since F($\mathcal{X}$)=1, a simple reasoning with lim sup gives the following equalities and the existence of its left members:

$$\lim_w \|\mathcal{X}_w\| = \|\mathcal{X}\|,$$
$$\lim_w f_n(\mathcal{X}_w) = f_n(\mathcal{X}),$$
$$\lim_w f_n(-\mathcal{X}_w) = f_n(-\mathcal{X}),$$

for every $n \in \mathbb{N}$.
Fix $\varepsilon > 0$. By the proof of Proposition (2.1.2) (iii) there exist $n \in \mathbb{N}$ and $U \in \tau$ such that $\mathcal{X} \in A_n \cap U$and diam $(\overline{A'_n} \cap U) \le \varepsilon$. In particular, as $A_n$is norm open then $f_n(\mathcal{X}) < 1$ so for w-large enough $f_n(\mathcal{X}_w) < 1$ and thus $\mathcal{X}_w \in \overline{A'_n}$.
Since for w-large enough we have $\mathcal{X}_w \in U$ we obtain $\|\mathcal{X}_w - \mathcal{X}\| \le \varepsilon$.This shows that the net $(\mathcal{X}_w)$ converges to $\mathcal{X}$ in norms, so the norm topology and τ coincide on S.

Clearly the in statement (ii) of the preceding theorem can be replaced by any constant greater 1. In fact every function of the form $\|\cdot\| + aF$ with $a > 0$ has the same property. This also shows that the norm can be approximated uniformly by functions with the kadec property provided at least one such function exists.
Note that S is a norm G_δ-set in $B = \{\mathcal{X} \in X : F(\mathcal{X}) \le 1\}$, $thus$ (S, τ) is completely metrizable.
A remarkable theorem of Kadec (see [45]) shows that every separable Banach space has an equivalent τ-Kadec norm for the topology τ of convergence on a fixed quasi-norming subset of its dual space. The following result extending Kadec's theorem.
Theorem (2.1.14)[43]: Let X be a Banach space and τ a weaker topology such that $\bar{B}'_X$ is bounded. Then X has an equivalent τ-Kadec norm if and only if X has $P(\|\cdot\|, \tau)$ where the sets (A_n) in Definition (2.1.1) are convex, in other words, if there exist convex, sets $A_n \subset X$ such that for every $\mathcal{X} \in X$ and every $\varepsilon > 0$there are $n \in \mathbb{N}$ and $U \in \tau$ such that $\mathcal{X} \in A_n \cap U$ and diam$(A_n \cap U) < \varepsilon$.
Proof: If we begin with (A_n) convex in the proof of Theorem(2.1.13) it is easily checked that all the families of sets built there are still convex. Thus F is subadditive and so it is an equivalent τ-Kadec norm.
For the converse assume that the norm of X is τ-Kadec. The proof of Lemma (2.1.11) shows that X has $P(\|\cdot\|, \tau)$ with a sequence of differences of closed balls centered at 0. As the closed balls are τ-closed we deduce that the sequence of closed balls with rational radii satisfies what is required.
We do not know if property $P(\|\cdot\|, w)$ implies the existence of anequivalent Kadec norm.

Molto, Orihela and Troyanski [62] have given a characterization of the existence of an equivalent LUR norm in a Banach space using a variant of Definition (2.1.3). Their result can be reformulated in similar terms to those of Definition(2.1.1) as follows: a Banach space X admits a LUR norm if and only if there exists a sequence of sets $A_n \subset X$ such that for every $\mathcal{X} \in X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and an open semispace U such that $\mathcal{X} \in A_n \cap U$ and diam$(A_n \cap U) < \varepsilon$. Note that the topological counterpart of this result is Theorem(2.1.13) applied to the weak topology but to deduce that the function F is in fact a Kadec norm we did need a geometric assumption about the sets A_n.

A Banach space X is said to be weakly countably determined (WCD) if there exists a sequence (K_n) of w^*-compact subset of X^{**} such that for every $x \in X$ and every $\mathcal{Y} \in X^{**} \setminus X$

there is $n \in \mathbb{N}$ $with\ \mathcal{X} \in K_n\ and\ \mathcal{Y} \notin K_n$. WCD Banach spaces generalize in a natural way the weakly compactly generated Banach spaces (WCG), that is, the spaces containing a total weakly compact set. A WCD Banach space admits a LUR norm [71]
The coincidence of Borel families in the following theorem imshows one by Talagrand [68] for subspaces of WCG Banach spaces.
Theorem (2.1.15)[43]: Let X be a WCD Banach space and let τ be a Hausdorff vector topology weaker than the weak topology of X. Then X has $P(\|\cdot\|, \tau)$. Moreover, if $\bar{B}_X^\tau$ is bounded then X also admits a τ-Kadec norm topology and

$$\mathrm{Borel}(X, \|\cdot\|) = \mathrm{Borel}(X, \tau).$$

Proof: We can assume without loss of generality that the sequence (K_n) is closed under finite intersections. We claim that the sequence of w^*-closed convex hulls $\{\overline{co(K_n)}^{w^*}\}$ also satisfies the above definition. Indeed, fix $\mathcal{X} \in X$ and $\mathcal{Y} \in X^{**} \setminus X$. The set $K = \bigcap_{\mathcal{X} \in K_n} K_n$ is a weakly compact set of X containing $\mathcal{X}$. Now, since $\overline{co(K)}^w$ is a weak*-compact convex set not containing $\mathcal{Y}$, there is a weak*-open half space H such that $\mathcal{X} \in H$ and $\mathcal{Y} \notin \bar{H}^{w^*}$. By compactness, there is $n \in \mathbb{N}$ such that $\mathcal{X} \in K_n \subset H$. As $\overline{co(K_n)}^{w^*} \subset \bar{H}^{w^*}$ we see that $\mathcal{X} \in \overline{co(K_n)}^{w^*}$ and $\mathcal{Y} \notin \overline{co(K_n)}^{w^*}$. This ends the proof of the claim.

First we check that X has $P(w, \tau)$. For every $\mathcal{X} \in X$ define

$$S_{\mathcal{X}} = \bigcap_{K_n \ni \mathcal{X}} K_n$$

By definition of WCD it is clear that $S_{\mathcal{X}}$ is a weakly compact subset of X.

If we take $\{S_{\mathcal{X}}\}$ as Σ and the traces on X of finite intersections of K_n's as a countable family Π, then the conditions in Lemma (2.1.10) are satisfied. Indeed, Σ covers X, and τ and w coincide on every $S_{\mathcal{X}}$ by compactness, so Σ is good for (w, τ). Now let W be a weak neighborhood of 0 and let W^1 be a weak* neighborhood of 0 in X^{**} such that $W = X \cap W^1$. For some increasing sequence (n_j) of integers we have $S_{\mathcal{X}} = \bigcap_j K_{n_j}$. By compactness there are a finite number of K_{n_j}'s whose intersection is contained in $S_{\mathcal{X}} + W^1$. So X has convex $P(w, \tau)$.

Since a WCD Banach space admits a Kadec norm, it has convex $P(\|\cdot\|, w)$. Now X has $P(\|\cdot\|, \tau)$ by Proposition (2.1.2) (ii) with convex sets.
The existence of a τ-Kadec equivalent norm follows from Theorem (2.1.14), and the coincidence of Borel sets follows from Corollary (2.1.6).

Using the general definition of a countably determined topological space (X, τ_1) in terms of use maps one can show that X has $P(\tau_1, \tau_2)$ for every weaker Hausdorff topology τ_2, but it is not clear if that implies the coincidence of Borel sets. For example, in the preceding theorem, if we want to show the coincidence of Borel sets for τ and the weak topology directly from the fact that X has $P(w, \tau)$ we have to check that $X \cap K_n$ is τ-Borel, which is not evident except in the case of a WCG space. Roughly speaking that was the argument of Talagrand [68], but WCD spaces were introduced some years later.

In the particular case of a dual WCD space, when τ is the weak* topology it is known that the space admits an equivalent dual LUR norm [8]. Without the hypothesis of WCD the result may not be true: the space $J(w_1)$ is a dual with the Radon-Nikodym property, so it admits an equivalent LUR norm [45], but Borel

$(J(w_1), w^*)$is a proper sub set of Borel $(J(w_1), w)$ = Borel $(J(w_1), \|\cdot\|)$ (see [50]). A natural generalization of dual WCD is the dual spaces X^* such that $(B_{X^{**}}, w^*)$ is a Corson compact set but in this case there may be no dual LUR norm [55].

The next corollary is inspired by a result of [48] for WCG spaces.

Corollary (2.1.16)[43]: let Y be a Banach space and τ a vector topology weaker than the weak topology of Y such that the unit ball $\bar{B}_Y^\tau$ is bounded. If X is a WCD norm closed subspace of Y then X is a τ-Borel set in Y.

Proof: Note that τ is Hausdorff. We deduce from Theorem (2.1.15) that X has $P(\|\cdot\|, \tau)$.Now apply Proposition (2.1.5) (b).

It is not difficult to see that under the conditions of Corollary (2.1.12) if X is $K_{\sigma\delta}$ in (X^{**}, w^*) (for example if X is WCG) then it is an $F_{\sigma\delta}$ in (Y, τ) while the proof of Corollary(2.1.16) shows that X is an$(F \cap G)_{\sigma\delta}$. It is not known if a WCD Banach space is always a $K_{\sigma\delta}$ in (X^{**}, w^*)(see [47]).

It is known that K-analytic topological spaces are Cech-analytic for every Hausdorff weaker topology. The same result is not true in general for WCD topological spaces. The next corollary gives a positive answer in the particular case of Banach spaces and "reasonable" topologies.

Corollary (2.1.17)[43]: Let X be a WCD Banach space and τ the topology of convergence on a quasi-norming subset of X^*. Then (X, τ) is $\tilde{C}$ech-analytic.

Proof: Using an equivalent norm we can suppose that τ s given by a norming subset. Then apply Corollary (2.1.7).

Let us mention here that it is a consequence of Proposition (2.1.4) and Theorem(2.1.15) that under the hypothesis of Corollary (2.1.17) ,(X, τ) is σ-fragmentable and, in particular, the τ-compact subsets of X are fragmentable (see [46]).

A typical situation is the case of $C(K)$ spaces with the pointwise topology. There is a huge family of compact spaces K called Valdivia compact sets such that $C(K)$ admits a LUR norm which makes the unit ball pointwise closed [70]. So the results above are applicable, in particular the Borel sets for the norm and pointwise topologies coincide. Recently Haydon, Jayne, Namioka and Rogers [56] have shown that if K is a totally ordered set that is compact in its order topology then $C(K)$ admits a norm with the Kadec property for the pointwise topology so the same coincidence of Borel sets holds.

A different class of compact spaces where we can check directly the coincidence of Borel sets in $C(K)$ for the weak and pointwise topologies is the class of Radon-Nikodym compact spaces. Originally, a compact space is called Radon-Nikodym when it is homeomorphic to a w^*-compact subset of a dual with the Radon-Nikodym property. Equivalently a compact set K is Radon-Nikodym if and only if there exists a stronger lower semicontinuous metric d on K such that every Radon measure on K is the restriction of a Radon measure on (K, d) [64] and [57].

Theorem (2.1.18)[43]: Let K be a Radon-Nikodym compact space. Then C(K) has an equivalent point wise lower semi continuous norm such that on its unit sphere the weak and point wise topologies coincide, C(K) has $P\big(w, t_P(K)\big)$and

$$\text{Borel}(C(K), w) = \text{Borel}\big(C(K), t_P(K)\big).$$

Proof: A continuous function on K is d-uniformly continuous. Indeed, suppose not. Then we can take sequence $(\mathcal{X}_n)$ and $(\mathcal{Y}_n)$ in K such that $\lim_n d(\mathcal{X}_n, \mathcal{Y}_n) = 0$ while$|f(\mathcal{X}_n) - f\,(\mathcal{Y}_n)| \geq \delta\ for$ some $\delta > 0$. By taking an ultrafilter we make the sequences converge to the limits $\mathcal{X}$ and $\mathcal{Y}$ respectively. But by the lower semicontinuity of d we have $d(\mathcal{X}, \mathcal{Y}) = 0\ so\ \mathcal{X} = \mathcal{Y}$ and this contradicts the continuity of *f*.

Fix a d-dense set $(\mathcal{X}_n)_{\alpha\in\Gamma}$. Now we define the seminorms O_n as follows:

$$O_n(f) = \sup_{\alpha} \sup\{|f(\mathcal{X}) - f(\mathcal{X}_\alpha)| : d(\mathcal{X}, \mathcal{X}_\alpha) \leq 1/n\}.$$

Clearly O_n is pointwise lower semicontinuous and since every $f \in C(K)$ is d-uniformly continuous, for every $\delta > 0$ there exists $n \in \mathbb{N}$ such that $O_n(f) < \delta$.

Define a new norm by the formula

$$|||f||| = \|f\| + \sum_{n=1}^{\infty} \frac{1}{2^n} O_n(f).$$

Evidently $\|\cdot\| \leq |||\cdot||| \leq 3\|\cdot\|$. Thus $|||\cdot|||$ is an equivalent norm in C(K).

It is also not hard to check the unit ball of $|||\,.\,|||$ is pointwise closed.

We now check that the weak and pointwise topologies coincide on $S = \{f \in C(K) : |||f||| = 1\}.\ Let\ (f_w)$ be a net in S pointwise converging to $f \in S$. Take a Radon measure μ with $\|\mu\| \leq 1$ that we suppose already defined on Borel(K, d) and take $\varepsilon > 0$.

From the pointwise lower semicontinuity of $\|\cdot\|$ and O_n, reasoning as in Theorem (2.1.13) we deduce that $\lim_w O_n(f_w) = O_n(f)$ for every $n \in \mathbb{N}$.

Now fix $n \in \mathbb{N}$ such that $O_n(f) \leq \varepsilon/8$. Then for w large enough $O_n(f_w) \leq \varepsilon/6$. Since μ has a d-separable d-support we can fix $F \subset \Gamma$ finite such that

$$|\mu|\left(\bigcup_{\alpha\in F} B[\mathcal{X}_{\alpha 1} 1/n]\right) > |\mu|(K) - \frac{\varepsilon}{4}.$$

If ω is large enough then $|f_\omega(\mathcal{X}_\alpha) - f(\mathcal{X}_\alpha)| \leq \varepsilon/6$ for α ∈ F. So $|f_\omega(\mathcal{X}) - f(\mathcal{X})| \leq \varepsilon/2$ for every $\in \bigcup_{\alpha\in F} B(\mathcal{X}_\alpha, 1/n)$.

If we have in mind that $\|f\|$ and $\|f_\omega\|$ are bounded by 1, an easy calculus gives

$$|\mu(f_\omega - f)| \leq \int |f_\omega - f| d|\mu| \leq \varepsilon,$$

which implies that (f_ω) converges weakly to *f*.

Now apply Lemma (2.1.11) to deduce that C(K) has $P(w, t_P(K))$. Since the unit ball is pointwise closed the weak and pointwise topologies have the same Borel sets by Proposition (2.1.2)(iv); moreover, every weakly open set is a countable union of differences of pointwise closed sets.

Clearly Theorem (2.1.15) is still true for a continuous image of a Radon-Nikodym compactum. We know no example of a compact space with different Borel sets for the weak and pointwise topologies.

Note that if K is Radon-Nikodym compact and $(C(K), w)$ has $\|\cdot\|$-SLD, then $(C(K), t_P(K))$ has $\|\cdot\|$-SLD. In particular, K has Namioka property (see [58].)

Corollary (2.1.19)[260]: A τ^2-Kadec norm $\|\cdot\|$is τ^2-lower semi continuous, that is, its unit ball is always τ^2-closed.

Proof: Suppose that $\|\cdot\|$is not τ^2-Isc. Then there is a net $(\mathcal{X}_{w^2})$ on the unit sphere S_X and a point $\mathcal{X}$ outside the unit ball B_X such that τ^2-$\lim_{w^2} \mathcal{X}_{w^2} = \mathcal{X}$ take numbers $t_{w^2} > 1$ such that $\|\mathcal{X} + t_{w^2}(\mathcal{X}_{w^2} - \mathcal{X})\| = \|\mathcal{X}\|$. Let $\mathcal{Y}_{w^2} = \mathcal{X} + t_{w^2}(\mathcal{X}_{w^2} - \mathcal{X})$. Note that $\{t_{w^2}\}$ is bounded because $\inf_{w^2}\|\mathcal{X}_{w^2} - \mathcal{X}\| > 0$. We deduce that τ^2-$\lim_{w^2} \mathcal{Y}_{w^2} = \mathcal{X}$. Since $\|\mathcal{Y}_{w^2}\| = \|\mathcal{X}\|$ we should have $\lim_{w^2}\|\mathcal{Y}_{w^2} - \mathcal{X}\| = 0$, but this is impossible because $\|\mathcal{Y}_{w^2} - \mathcal{X}\| \geq \|\mathcal{X}_{w^2} - \mathcal{X}\|$.

Section (2.2): Function Spaces with Weak Topology

One of the main results is that the duality $\langle \ell^\infty, (\ell^\infty)^* \rangle$ is not Borel; see Corollary (2.2.2) for a precise statement.

We shall derive this fact from the more general Theorem (2.2.1) concerning the Banach spaces C(K) of real-valued continuous functions on compact F-spaces. A compact space K is an F-space if any continuous map $c: U \to [0, 1]$ defined on an open σ −compact set in K can be continuously extended over K: cf, [77] .We shall write $C_w(K)$ when considering the function space with the weak topology. The result we are just about to state involves C-measurability, a notion essentially weaker than Borel measurability. The C-sets in a topological space are the elements of the smallest $\sigma - algebra$ containing all open sets and closed under the Souslin operation $\mathcal{A}; cf.$ [82]. A function $f: X \to Y$ is C-measurable if $f^{-1}(U)$ is a C-set in X provided that U is open in Y.

Theorem (2.2.1)[72]: For each infinite compact F-space K, the evaluation map $e: K \times C_w(K) \to \mathbb{R}, e(x, f) = f(x),$ is not C-measurable.

Corollary(2.2.2)[72]: The duality map $\langle , \rangle: (\ell^\infty, \text{weak}) \times ((\ell^\infty)^*, \text{weak}^*) \to R, \langle x, x^* \rangle = x^*(x)$,is not C-measurable. To derive this result from Theorem (2.2.1). Let us identify ℓ^∞ with $C(\beta\text{N}), \beta\text{N}$ being the Čech-Stone compactification of the natural numbers N. then upon identification of $x \in \beta\text{N}$ with the probability measure supported by $\{x\}$, one can consider βN as the subspace of $((\ell^\infty)^*, \text{weak}^*)$, and the evaluation map $C(\beta\text{N}) \times \beta\text{N} \to \mathbb{R}$ is the restriction of the duality map $\langle , \rangle$.

It is worth noticing that a theorem of Rosenthal [89] asserts that under the continuum hypothesis, ℓ^∞ embeds in C(K) for any infinite F-space K. In fact, assuming the continuum hypothesis. Theorem (2.2.1).and Corollary (2.2.2) are closely related to each other; cf. [75].

Page (2):

We shall prove a slightly more refined version of Theorem (2.2.1) , considering a topology τ in C(K) which, on norm-bounded sets in C(K), is between the weak and norm topologies (the topology will play an essential role .The idea of the proof of Theorem (2.2.1) is closely related to the reasoning by Jayne. Namioka and Rogers [80] (cf, also [81], to the effect that the spaces C(K) in Theorem (2.2.1) are not σ −fragementable. They proved a stronger theorem that "tree-complete" spaces K have function spaces which are not σ − fragementable. The idea of Jayne, Namioka and Rogers can also be adapted in our case, by a refinement of the proof of Theorem(2.2.1) applied to a tree-complete compact space K defined by Haydon and Zizler [79], it yields C(K) without any subspace isomorphic to ℓ^∞ and non-Borel evaluation map $e: K \times C(K) \to \mathbb{R}$.

Talagrand [69] proved that Borel $\sigma -$ algebras inℓ^∞ associated with the weak and the norm topologies differ, that is, Borel $(\ell^\infty, \text{weak})$ ≠Borel$(\ell^\infty, \text{norm})$; cf [90]. However, using a certain result oHaydon [78] concerning function spaces on trees, one can define a compact (scattered) space K with Borel(C(K), weak), ≠ Borel (C(K), norm) and the evaluation map $e: K \times C_w(K) \to \mathbb{R}$ Borel-measurable; cf.

The reader is referred for some links between the topics discussed in the paper and interesting recent work by Burke [73], and Kendrov , Kortezov and Moors [83], [84].

Let c(K) be the function space on a compact non-scattered space K. Let us fix an irreducible continuous $\emptyset: Z \to [0,1]$ from a compact subset $Z \subseteq K$. (Since K is not scattered, there is acontinuous surjection $u : K \to [0,1]$; let Z be a minimal compact set mapped bu u onto $[0,1] and\ \emptyset = u|z\cdot$) Let $\mathfrak{T}$ be the collection of all sets of the form $A \cup \emptyset^{-1}(B), where\ A \subseteq K, Z$ is compact, B is closed in $[0,1]$ and $\emptyset^{-1}(B)$ has relatively empty interior in Z. We shall consider in C(K) a topology τ associated with $\emptyset$, generated by basic sets.

$$N(f, C) = \{g \in C(K): g|c = f|c\}, \qquad C \in \mathfrak{T}. \tag{1}$$

Let $C_\tau(K)$ be the function space C(K) equipped with the topology τ.

$$c: U \to [0,1], \text{where U=dom c is open and } \sigma - \text{compact}. \tag{2}$$

We shall consider $\wp$ with the discrete topology, and let $\wp^{\mathbb{N}}$ be the countable product of $\wp$.

Let

$$\mathcal{M} = \{(c_1, c_2, \cdots) \in \wp^{\mathbb{N}} : \overline{domc_\iota} \subseteq dom\ c_{i+1}, c_{i+1}|domc_i = c_i\}. \tag{3}$$

Let

$$\mathcal{Y} \subseteq Z \times C_\tau(K) \times \mathcal{M} \tag{4}$$

Be the subspace of the product consisting of all sequences

$$y = (x, f, c_1, c_2, \cdots) \tag{5}$$

Such that (cf. (3))

$$x \in Z \setminus \bigcup_{i=1}^{\infty} domc_i, \qquad 0 \le f \le 1, \tag{6}$$

$$(c_1, c_2, \cdots) \in \mathcal{M}\ \ and\ f|domc_i \qquad for\ all\ i \in N. \tag{7}$$

Lemma (2.2.3)[72]: Let $\mathcal{g}_1, \mathcal{g}_2, \cdots$ be open sets in $\mathcal{y}$, dense in a nonempty open set $\mathcal{U}$ in $\mathcal{y}$. Then there are points $y_1 = (x_i, f, c_1, c_2, \cdots) \in \cap_n, \mathcal{k}_n, i = 0, 1$, such that $f(x_0) = 0$, $f(x_1) = 1$.

Proof: To begin, let us introduce convenient nation for basic open sets in the space $\mathcal{Y}$. For every finite sequence $(c_1, c_2, \cdots, c_r)$ with $\overline{domc_\iota} \subseteq dom\ c_{i+1}, c_{i+1}|domc_i = c_i$ (cf. (3)), let

$$N(c_1, \cdots, c_r) = \{(c_1, \cdots, c_r, c_{r+1}, \cdots): (c_1, c_2, \cdots) \in \mathcal{M}\}. \tag{8}$$

Then basic open sets in $\mathcal{y}$ are of the form.

$$N(U, f, C, c_1, \cdots, c_r) = [U \times N(f, C) \times N(c_1, \cdots, c_r)] \cap \mathcal{y}, \tag{9}$$

Where N(*f*, C) and $N(c_1, \cdots, c_r)$ are defined in (1) and (8),

$$U \subseteq K \text{ is open and } \quad U \cap domc_r = \phi. \tag{10}$$

Note that the set in (9) is nonempty if and only if (cf. (4) and (7))

$$U \cap Z, \overline{domc_r} \ne \phi\ \ and\ f|domc_r = c_r. \tag{11}$$

Now, in $\mathcal{Y}$, find a nonempty basic open set

$$\mathcal{U}_0 = N(U_0, f_0, C_0, c_1, \cdots, c_{r0}) \subseteq \mathcal{g}_0. \tag{12}$$

We shall choose inductively basic open sets.

$$\phi \ne \mathcal{U}_n = N(U_n, f_n, C_n, c_1, \cdots, c_{rn}) \subseteq \mathcal{U}_{n-1} \cap \mathcal{g}_n,\ n - 1,2,\cdots \tag{13}$$

Such that

$$\overline{U_{n+1}} \subseteq U_n, \quad C_n \subseteq C_{n+1}, \tag{14}$$
$$C_n \subseteq \text{dom}c_{rn}, \qquad n \geq 1, \tag{15}$$

and there are points

$$a_n, b_n \in domc_{rn} \cap U_{n-1}, \quad c_{rn}(a_n) = 0, \quad c_{rn}(b_n) = 1, \ n \geq 1 \tag{16}$$

To this end, let $\mathcal{U}_0$ be as in (12) and assume that $\mathcal{U}_n$ has already defined Since $\mathcal{U}_n \cap \mathcal{g}_{n+1} \neq \phi$, there is a basic open set in $\mathcal{Y}$ such that :

$$\phi \neq N(U, f, C, c_1, \cdots, c_{r_n}, c_{r_n+1}, \cdots, c_r) \subseteq \mathcal{U}_n \cap \mathcal{g}_{n*1}. \tag{17}$$

One can assume that $U \subseteq U_n \ \ and \ C_n \subseteq C$.

We set $C_{n+1} = C$. Since $U \cap Z \neq \phi$ (cf. (11)), and $C \cap Z$ is nowhere dense in Z (cf, the definition of $\mathfrak{T}$), there is an open set W in K with $W \cap Z \neq \phi$, and $\overline{W} \subseteq U \setminus C$. Let W_0, W_1 be disjoint nonempty open set in K, $\overline{W_0 \cup W_1} \subseteq W, W \cap Z \setminus (\overline{W_0 \cup W_1}) \neq \phi, \ and \ let \ a_{n+1} \in W_0, b_{n+1} \in W_1. Let \ f_{n+1}: K \to [0,1]$ be a continuous function which coincides with f on $(K, W), f_{n+1}(a_{n+1}) = 0, f_{n+1}(b_{n+1}) = 1.$ Finally, let H be an open $\sigma-$ compact set containing(K \ W) ∪ $(\overline{W_0 \cup W_1})$ with $Z \setminus \overline{H} \neq \phi$, and let us set $U_{n+1} = k \ / \ \overline{H}$, $r_{n+1} = r + 1$, and declare $c_{r_{n+1}}$ to be the restriction of f_{n+1} to H.

Having defined the sets $\mathcal{U}_n$ in (13), let us consider a continuous function $f: K \to [0,1]$ extending all c_r. Since K is an F-sp ace, (2) and (3) guarantee the existence of such an extension. Next, using (16) and (14), let us pick.

$$x_0 \in \bigcap_n U_n \cap \overline{\{a_n : n \in N\}}, \qquad x_1 \quad \in \bigcap_n U_n \cap \overline{\{b_n : n \in N\}}, \tag{18}$$

We claim that

$$y_i = (x_i, f, c_1, c_1, \cdots) \in \bigcap_n \mathcal{g}_n, \qquad i = 0, 1. \tag{19}$$

To this end , we shall make sure that $y_0, y_1 \in \mathcal{U}_n$ for every *n* ;cf. (13). Indeed, $x_{i \in \mathcal{U}_n}$ (cf. (16) and (20)), and $(c_1, c_2, \cdot\cdot\cdot)$ extends $(c_1, \cdot\cdot\cdot, c_{r_{n+1}})$. Finally, *f* and f_n coincide on C_n (cf. (15) and (11)). In effect, $f \in N(f_n, C_n)(cf. (1))$, and hence $y_i \in \mathcal{U}_n$ (cf (9)). Moreover, $f(a_n) = 0, f(b_n) = 1 \ \ for \ n \in N$ (cf. (16)), and therefore, by (20), $f(x_i) = i, \ i = 0, 1$.That concludes the proof of the lemma.

Proposition (2.2.4)[72]: Let K be an infinite F-space and let τ be the topology in C(K) generated by basic sets (1). Then the evaluation map $e: K \times C_\tau(K) \to \mathbb{R}$ is not C-measurable.

To see that Theorem(2.2.1) follows from Proposition(2.2.4), let us make the following observation.

We devoted to a proof of Proposition (2.2.4). Let $\wp$ be the collection of continuous functions.

Proof: We will now derive Proposition (2.2.4) from Lemma(2.2.3) let

$$H = \{(x, f) \in Z \times C_\tau(K): f(x) > 0\}. \tag{20}$$

Let $n: \mathcal{y} \to Z \times C_\tau(K)$ be the restriction to $\mathcal{y}$ of the projection parallel to $\mathcal{M}$, and let

$$\mathcal{H} = \pi^{-1}(H) = \{(x, f, c_1, c_2, \cdot\cdot\cdot) \in \mathcal{y}: f(x) > 0\}. \tag{21}$$

Aiming at a contradiction, assume that $\mathcal{H}$ is a C-set in $\mathcal{y}$; hence it is open modulo meager sets in $\mathcal{y}$; cf. [82], [29], [83] Lemma (2.2.3) shows, in particular, that $\mathcal{y}$ is a Baire space, hence either $\mathcal{H}$ or $\mathcal{y} \setminus \mathcal{H}$ is nonmeager in $\mathcal{y}$. In effect, there is a $G_\delta - set \ \wp$ in $\mathcal{y}$, dense in some nonempty open set in $\mathcal{Y}$ such that either $\wp \subseteq \mathcal{H}$ *or* $\wp \cap \mathcal{H} = \phi$. However, using Lemma(2.2.3) again, we conclude that $\wp$ interests both $\mathcal{H}$ and its complement; cf. (20). This contradiction ends the proof of the proposition.

Lemma (2.2.6)[72]: Let $\phi: S \to C_w(K)$ be a continuous map from a choquet space S of weight 2^{N_0} to the function space endowed with the weak topology such that ϕ takes nonempty open sets to sets of norm-diameter 1. Then there is a non-C-set in $C_w(K)$ which is norm-discrete.

Proof: The Choquet property of S provides a function σ associating to each finite sequence $U_1,\cdots,U_n$ of nonempty open in snonempty open set $\sigma(U_1,\cdots,U_n) \subseteq U_n$ such that

$$\bigcap_n U_n \neq \phi, \qquad \text{whenever } U_{n+1} \subseteq \sigma(U_1,\cdots,\mathcal{U}_n), n = 1,2,\cdots \quad (22)$$

Let us fix a base $\mathfrak{R}$ for S of cardinality 2^{N_0}, and let Λ be the collection of dyadic systems

$$\mathfrak{D} = \{T_t: t \in 2^{N_0}\},\ \phi \neq U_1 \in \mathfrak{R}, \quad (23)$$

where$U_{t'} \subseteq U_{t''}$ift' extends t'' and $U_{t'} \cap U_{t''} = \phi$ *if* $t', t'' \in 2^n, t' \neq t''$, and the following conditions are satisfied.

$$U_t \subseteq \sigma\left(U_{\frac{t}{1}}, U_{\frac{t}{2}},\cdots,\mathcal{U}_{\frac{t}{n}-1}\right), \qquad t \in 2^n, \quad (24)$$

$$\|\phi(u) - \phi(v)\| > \frac{1}{2}, \text{whenever } u \in U_{t'}, v \in U_{t''}, t', t'' \in 2^n, t' \neq t'' (25)$$

Page (6):

For each $\mathfrak{D} \in \Lambda$ and $t \in 2^N$ we pick $P_{\mathfrak{D}}(t) \in \bigcap_n U_{t/n}$ and let

$$A(\mathfrak{D}) = \phi(\{P_{\mathfrak{D}}(t): t \in 2^N\}). \quad (26)$$

Then, by(22), (24) and (25),

$$|A(\mathfrak{D})| = 2^{N_0} \text{and} \|f - g\| \geq \frac{1}{2} \text{ for distinct } f, g \in A(\mathcal{g}). \quad (27)$$

Since $|\mathfrak{B}| = 2^{N_0}$ *we see* $|\Lambda| = 2^{N_0}$, and we can list the elements of Λ *as* $\{\mathfrak{B}_\alpha: \alpha < 2^{N_0}\}$. Then, by (27), one can pick, by transfinite induction, pairwise distinct $f_\alpha, g_\alpha \in A(\mathfrak{D}_\alpha)$ such that $\|f_\alpha - f_\beta\| \geq \frac{1}{4}$for $\alpha > \beta$. Letting B $= \{f_\alpha: \alpha < 2^{N_0}\}$,we have .

$$B \cap A(\mathfrak{D}_\alpha) \neq \phi \neq A(\mathfrak{D}_\alpha) \backslash B,\ \ \alpha < 2^{N_0}. \quad (28)$$

Since B is norm-discrete it is enough to make sure that B is not a C-set in$C_w(K)$. Assume the contrary. Then $\phi^{-1}(B)$ is a C-set, and hence it is open modulo meager sets in the Baire space $\mathcal{Y}$.Therefore are nonempty open sets $G = \bigcap_{i=1}^{\infty} G_i$ is contained in either $\phi^{-1}(B)$ or its complement.

Notice that if $\|f - g\| > \frac{1}{2}, f, g \in C(K)$, *there is* $a \in K$ *with* $|f(a) - g(a)| > \frac{1}{2}$ and hence there are neighborhoods U, V of f and g in $C_w(K)$ such that $\|f' - g'\| > \frac{1}{2}$ whenever $f' \in U, g' \in V$. Using this fact and the assumption that ϕ takes nonempty open sets to sets with norm-diameter 1, one readily constructs a dyadic system $\mathfrak{D}$ (cf. (23)) satisfying (24)and (25) and $U_1 \subseteq G_n$for t $\in 2^n$. Then (cf. (26)) $A(\mathfrak{D})$ is either contained in B or disjoint from B. However ,$\mathfrak{D} = \mathfrak{D}_\alpha$, for some α, and we have reached a contradiction with (36).

Theorem (2.2.5)[72]:for any infinite compact F-space K there is a norm-discrete set in C(K) which is not a C-set with respect to the weak topology.

We shall derive this theorem from a lemma, closely related to Fremlin's theorem 7J in [76]. We refer to Kechris [82] for the notion of Choquet spaces (or in the terminology of [76], weakly-α-favorable spaces).

Proof. We first prove the theorem for F-spaces of weight 2^{N_0}, and then we reduce the general case to the case for weight 2^{N_0}. Let us recall that any infinite compact F-space contains a copy of βN and hence it has weight at least 2^{N_0}.

(A) Let K be a compact F-space of weight 2^{N_0}., we shall consider $C_\tau(K)$, the function space C(K) with the topology τ determined by basic sets.

$$N(f, C) = \{g \in C(K) : g|c = f|c\}, \quad C \in \mathfrak{T} \tag{29}$$

Since Z is the intersection of at most 2^{N_0} open sets, τ has a base of cardinality 2^{N_0} . Let $\mathcal{M}$ be the space defined in (3), and let S be the projection of the space $\mathcal{Y}$ defined parallel to the Z-coordinate, that is, let

$$S \subseteq C_\tau(K) \times \mathcal{M} \tag{30}$$

consist of all sequences

$$\sigma = (f, c_1, c_2, \cdots) \tag{31}$$

such that

$$Z \setminus \bigcup_{i=1}^{\infty} domc_i \neq \phi, \qquad 0 \leq f \leq 1. \tag{32}$$

$$(c_1, c_2, \cdots) \in \mathcal{M}, \qquad f|domc_i = c_i. \tag{33}$$

We shall apply Lemma(2.2.6) to the map

$$\Phi: S \to C_w(K), \quad \Phi(f, c_1, c_2, \cdots) = f. \tag{34}$$

The map Φ is continuous, τ being stronger than the weak topology on the unit ball Basic open sets in S are of the form.

$$N(f, C, c_1, \cdots, c_r) = [N(f, C) \times N(c_1, \cdots, c_r)] \cap S \tag{35}$$

(cf. (1) and (8). The set in (28) is nonempty if and only if (cf. (25) and (26))

$$Z, \overline{domc_r} \neq \emptyset \text{ and } f|domc_r. \tag{36}$$

Since, for a $C \in \mathfrak{F}, C \cap Z$ has empty interior relative to Z, (36) yields that $Z, \left(C \cup \overline{domc_r}\right) \neq \emptyset$, and since Z has no isolated points, the image of the set in (28) under Φ has norm-diameter 1. Therefore, to apply Lemma(2.2.6).it is enough to make sure that the space S is Choquet. Playing the choquet game in S, (α) and (β) may restrict their moves to basic sets given in (28), and a winning strategy for (α) is to respond to a move $N\left(f_n, C_n, c_{r_n}\right) of\ (\beta), n$ even, in the following way. By the observation following (29),$Z\setminus\left(C_r \cup \overline{domc_r}\right) \neq \emptyset$, and (α) picks an open σ-compact $U \subseteq K$ containing $C_r \cup \overline{domc_r}$ with $Z\setminus\overline{U} \neq \emptyset$, sets $r_{n+1} = r_n + 1, c_{r_{n+1}} = f_n|U, f_{n+1} = f_n, C_{n+1} = C_n$, and plays with $N(f_{n+1}, C_{n+1},\ c_1, \cdots, c_{r_{n+1}})$. Since K is an F-space, there is $f \in \mathrm{C(K)}, 0 \leq \mathrm{f} \leq 1$, coinciding with every $c_i$on its domain. For each even $n, C_n \subseteq dom\ c_{r_{n+1}}\ and\ c_{r_{n+1}}$ coincides with $f_n$onC_n; $hence\ f \in N(f_n,\ C_n)$ and in effect $f \in \bigcap_{k=1}^{\infty} N(f_k, C_k)$. Moreover, $Z \setminus dom\ c_k \neq \emptyset$ for each k, and hence $Z \setminus \bigcap_{k=1}^{\infty} dom\ c_k \neq \emptyset$.It follows that $(f, c_1, c_2, \cdots) \in \bigcup_{K=0}^{\infty} N\left(f_n, C_n, c_1, \cdots, c_{r_n}\right)$, that is, (x) indeed wins in the game.

An application of Lemma (2.2.6) now completes part (A) of the proof.

(B) Let K be an arbitrary infinite compact F-space. Let us fix a uniformly Closed sub-algebra $\mathcal{A}_0$of C (K) containing the unit, with infinite linear dimension and $|\mathcal{A}_0| \leq 2^{N_0}$, and let us define, by transfinite induction, uniformly closed sub-algebras of C(K)

$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \mathcal{A}_\alpha \subseteq \cdots, \qquad \alpha < \omega_1, |\mathcal{A}_\alpha| \leq 2^{N_0}. \tag{37}$$

such that for any $f \in \mathcal{A}_\alpha$ there is $\hat{f} \in \mathcal{A}_{\alpha+1}$ with $\hat{f}(x) = 0\ if\ F(x) < 0$ and

$$\hat{f}(x) = 1\ if\ f(x) > 0. \tag{38}$$

If$\mathcal{A}_\alpha$ is defined, and $f \in \mathcal{A}_\alpha$, we have $\overline{\{x: f(x) < 0\}} \cap \overline{\{x: f(x) > 0\}} = \emptyset$,
K being an F-space; let us fix $\hat{f} \in C(K)$ as in (38). Then $\mathcal{A}_{\alpha+1}$ is the uniformly closed sub-algebra of C(K) generated by the set $\mathcal{A}_\alpha \cup \{\hat{f}: f \in \mathcal{A}_\alpha\}$.

The algebra $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ is uniformly closed in C(K). Let $q: K \to L$ be the question map identifying the points in K which are not separated by any function in $\mathcal{A}$.The adjoint map $q^*: C(L) \to \mathcal{A},\ \ q^*(u) = u^\circ q$, identifies the algebras. The compact space L is infinite ($\mathcal{A}_0$being infinite-dimensional), and the weight of L is not greater than 2^{N_0}, as $|\mathcal{A}| \leq 2^{N_0}$; cf. (37). Let us check that L is an F-space. By [77], it is enough to show that for any pair U, V of disjoint open $\sigma -$ compact sets in L, $\bar{U} \cap \bar{V} = \emptyset. Let\ u: L \to [-1, 1]$ be a continuous map with $u(x) < 0$ for $x \in U, u(x) > 0$ for $x \in V$. Then $f = q^*(u) \in \mathcal{A}$, and hence $f \in \mathcal{A}_\alpha$ for some α; let $\hat{f} \in \mathcal{A}_{\alpha+1}$ be as in (38). Let$\hat{f} = q^*(v)$. Then $v: L \to [0, 1]$ is zero on U and 1 on V. and $\bar{U} \cap \bar{V} = \emptyset$.

Now, by part (A), there is a norm-discrete set E in C(L) which is not C-set in the weak topology, and the set $q^*(E)$ has corresponding properties in C(K). Function space C(K) with different norm-Borel and weak-Borel sets and the Borel duality map for any compact scattered space K, the space $C(K)^*$ dual to C(K) can be identified with the space $\ell_1(K)$ of functions.

$$\lambda: K \to \mathbb{R}, \qquad \sum_{x \in K} |\lambda(x)| < \infty. \tag{39}$$

where the duality map is defined by

$$\langle f, \lambda \rangle = \sum_{x \in K} f(x). \quad \lambda(x). \tag{40}$$

Example (2.2.7)[72]: There exists a compact scattered space K such that the duality map $\langle , \rangle: C_w(K) \times (\ell_1(K), \text{weak}^*) \to \mathbb{R}$ is Borel, while Borel (C(K), weak) ≠ Borel (C(K), norm).

The space K in Example(2.2.7) , the one-point compactification of a standard tree, was considered by Haydon [54] in his work on reforming in function spaces. We shall begin with explaining some notions concerning the trees, following Deville, Godefroy and Zizler [74]

A tree T is a set with a partial order $\prec$ such that the segments $\{s: s \prec t\}$are well-ordered. We assume that if $\{s: s \prec t_1\} = \{s: s \prec t_2\}$
and the segments have a limit ordinaltype, then $t_1 = t_2$. We assume also that T has the least element$\langle \phi \rangle$, the roof. The topology of T is generated by the base consisting of the intervals $(s, t] = \{u: s \prec u \preccurlyeq t\}$ and the root is isolated. Let $\widehat{T} = T \cup \{\infty\}$ be the one-point compactification of T. The intervals $[s, t] = (s, t] \cup \{s\}$, where s is a successor in T, are open and closed in $\widehat{T}$. We shall denote by Succ(t) the set of immediate successors of t.

We shall first establish the following.

Proposition (2.2.8)[72]: For any tree T. the duality map $\langle,\rangle: C_w(\hat{T}) \times (\ell_1(\hat{\mathbb{T}}), \text{weak}^*) \to \mathbb{R}$ is Borel measurable.

Given a compact scattered space K and a finite set D ⊆ ℝ containing 0, we set

$$C(K,D) = \{f \in C(K): f(K) \subseteq \mathrm{D}\}. \tag{41}$$

In the proof of Proposition (2.2.10), we shall use the following discretization lemma, proved (in a slightly different form) in [54], [78], [86].

Proof: We shall verify the assertion in a few steps, the sky being Claim A, given below. Let us begin with some preliminary observations. Given $\mathcal{A} \subseteq C(\hat{\mathbb{T}})$ or $\Gamma \subseteq \ell_1(\hat{\mathbb{T}})$, we shall denote by $(\mathcal{A}, w)$or (Γ, w^*) the subspaces of $(C(\hat{\mathbb{T}})$, weak), or $(\ell_1(\hat{\mathbb{T}}), \text{weak}^*)$, respectively. For any $\mathcal{A} \subseteq \hat{\mathbb{T}}$ *and* $\lambda \in \ell_1(\hat{T})$, we write.

$$|\lambda|(A) \sum_{t \in A} |\lambda(t)|, \quad \|\lambda\| = |\lambda|(\hat{\mathbb{T}}). \tag{42}$$

For each open $W \subseteq \hat{\mathbb{T}}$ and $r \geq 0$, the set

$$\{\lambda \in \ell_1(\hat{\mathbb{T}}): |\lambda|(W) > r\} \text{is open in } (\ell_1(\hat{\mathbb{T}}), W^*), \tag{43}$$

and the map

$$\lambda \to \lambda(\infty) \text{ is Borel on } (\ell_1(\hat{\mathbb{T}}), W^*). \tag{44}$$

(For the reader's convenience, more details on property (44).) Let

$$C_0\left(\hat{\hat{\mathbb{T}}}\right)\{f \in C(\hat{\mathbb{T}}): f(\infty) = 0\}, \quad \sum = \{\lambda \in \ell_1(\hat{\mathbb{T}}): \|\lambda\| = 1, \lambda(\infty) = 0\}. \tag{45}$$

The assertion of Proposition (2.2.8) follows easily, as soon as it is established that the duality map.

$$\langle,\rangle: (C_0(\hat{T}), W) \times (\textstyle\sum, W^*) \to \mathbb{R} \text{ is Borel.} \tag{46}$$

Indeed, let δ_∞ be the Driac measure concentrated at ∞ and let $\mathbb{R}\delta_\infty$ be the line spanned by δ_∞ . Let $\sigma(\lambda) = \lambda - \lambda(\infty), \delta_\infty$. By (42)-(44), the mappings $\lambda \longmapsto \sigma(\lambda), \lambda \longmapsto \|\sigma(\lambda)\|$ are Borel, and so is $\lambda \longmapsto \sigma(\lambda)/\|\sigma(\lambda)\| \in \sum$, defined for $\lambda \notin \mathbb{R}\delta_\infty$.

Therefore (cf.(44)) yields that the duality map

$$\langle f, \lambda \rangle = \|\sigma(\lambda)\|. \langle f - f(\infty), \frac{\sigma(\lambda)}{\|\sigma(\lambda)\|} \rangle + \langle f(\infty), \lambda \rangle$$

is Borel on $(C_w(\hat{\mathbb{T}}), \text{weak}) \times (\ell_1(\hat{\mathbb{T}}) \setminus \mathbb{R}\delta_\infty, w^*)$ and of course it is also Borel on $C(\hat{T}) \times \mathbb{R}\delta_\infty$. We may consult for some details omitted in this reasoning.

The rest of the proof will be devoted to a verification of (46).

(A)Let $D \subseteq \mathbb{R}$ be a finite set, $0 \in D$, and let (cf. (41) and (45))

$$C_0(\hat{\mathbb{T}}, D) = \{f \in C_0(\hat{\mathbb{T}}, D): f(\infty) = 0\}. \tag{47}$$

Claim A. There are Borel sets $\mathcal{H}_n$ in $(C_0(\hat{\mathbb{T}}, D), w)$covering$C_0(\hat{\mathbb{T}}, D)$such that for every $f \in \mathcal{H}_n$ and finite, $F \subseteq \mathbb{T}$ there is aneighborhood $\mathscr{r}$ of f in $(\mathcal{H}_n, w)$ and a neighborhood W of F in Twith g/w for any g ∈ $\mathscr{r}$.

Proof. We shall prove first the claim for D = {0, 1}. Let

$$\varepsilon = C_0(\hat{T}, \{0,1\}), \tag{48}$$

whereε is equipped with the weak topology. For $f \in \varepsilon$, we let

$$S(f) = \{t \in T: f(t) = 1 \text{ and either } t = \langle \phi \rangle \text{ or there is s} \in \mathrm{T} \text{ with } t \in Succ(s) and\ f(s) = 0\}. \tag{49}$$

Let $f \in \varepsilon$ and $A_f \subseteq T \times T$ be the finite set of pairs (s, t) with $t \in Succ(s), f(s) = 0$ } and $f(t) = 1$. Then

$$\mathcal{U}_f = \{g \in \varepsilon: g(s) = 0, g(t) = 1 \; for \; (s,t) \in \mathcal{A}_f \text{ and} g(\langle\phi\rangle) = f(\langle\phi\rangle)\}. \quad (50)$$

is a neighborhood *of* f in ε such that

$$S(f) \subseteq S(g) \qquad for \;\; any \;\; g \in \mathcal{U}_f. \qquad (51)$$

It follows from (51) that each set $\{f \in \varepsilon: |S(f)| \geq n\}$ is open in ε, and therefore each set.

$$\varepsilon_n = \{f \in \varepsilon: |S(f)| = n\} \text{is Borel in } \varepsilon . \qquad (52)$$

We shall check that the sets $\mathcal{H}_n = \varepsilon_n$ satisfy the assertion of Claim A.Let $f \in \varepsilon_n$ and let $F \subseteq T$ be finite. For each $t \in F$, we choose$t^* = t$, if t is isolated in T, and $t^* \prec t$ such that t^* is a successor and f is constant on $[t^*, t]$,, if t is not isolated. Let $F^* = \{t^*: t \in F\}$. We set $\mathscr{r} = \{ \; g \in \varepsilon_n$: gcoincides with f on $F \cup F^* \cup \{\langle\phi\rangle\}$ } $\cap \mathcal{U}_f$(cf. (52)). and $W = \cup\{[t^*, t]: t \in F\}$. Notice that (50) and (52) imply that $S(f) = S(g)$. We shall check that any $g \in \mathscr{r}$ coincides with f on W. Let $w \in [t^*, t], t \in F$. Assume that $f(t) = 1$. Then take value 1 on $[t^*, t]$ and $g(t^*) = g(t) = 1$. Let $t^* \prec t$.

If $g(w) = 0$, let s be the maximal element of $[t^*, t]$ with $g(s) = 0$.

Then, for $u \in Succ(s) \cap [t^*, t], u \in S(g) \setminus S(f)$, which is impossible.

Similarly, if $f(t) = 0$, and hence f takes value 0 on $[t^*, t]$, we infer that $g(w) = 0$ for $w \in [t^*, t]$.

We shall now consider the general case, $D = \{0, d_1, d_2, \cdots, d_p\} \subseteq \mathbb{R}$.

Let $r_i: D \to \{0, 1\}$ take d_1 to 1 and other elements of D to 0, and let $\sigma_i: (C_0(\hat{T}, D), w) \to (\varepsilon, w)$ be defined by $\sigma_i(f) = r_i \circ f; cf.$ (49). Then. arranging the sets $\bigcap_{i=1}^{P} \sigma_i^{-1}(\varepsilon_{n_i}), n_i \in N$, into a sequence $\mathcal{H}_1, \mathcal{H}_2, \cdots$, we get Borel sets satisfying the assertion of Claim A.

(B) With $\Sigma$and $C_0(\hat{T}, D)$ defined by (45) and (42), we shall check the following claim.

Claim B. The duality map $\langle , \rangle: (C_0(\hat{T}, D), w) \times (\Sigma, w^*) \to \mathbb{R}$ is Borel.

Lemma(2.2.9)[72]: Let K be a compact scattered space. For $i \in \mathbb{N}$, there are Borel sets $\mathcal{D}_i \; in \; C_w(K)$, finite sets$D_i \subseteq \mathbb{R}, 0 \in D_i$, real numbers $\delta(i) > 0$ and continuous maps $\Phi_i: (\mathcal{D}_I, \text{weak}) \to (C(K, D_I), \text{weak})$ such that the following hold.

(i) $\|f - \Phi_i(f)\| < \delta(i)$ and$\Phi_i(f)(x) = 0$ when $f(x) = 0$.

(ii) For any $f \in C(K) and \; \varepsilon \gg 0$, there is i with $f \in \mathcal{D}_i \; and \; \delta(i) < \varepsilon$.

Proof: To this end, let us consider the sets $\mathcal{H}_n$ defined in Claim A, and for $M > 0, r \in \mathbb{R}$, let us set.

$$\mathcal{H}(n, M, r) = \{(f, \lambda) \in \mathcal{H}_n \times \Sigma: \|f\| \leq M \; and \; \langle f, \lambda\rangle > r\} \qquad (53)$$

To get Claim B it is enough to verify that the sets $\mathcal{H}(n, M, r)$ are Borel. In fact, we shall see that $\mathcal{H}(n, M, r)$ is relatively open in the Borel set $\{(f, \lambda) \in \mathcal{H}_n \times \Sigma: \|f\| \leq M\}. Let \; (f, \lambda) \in \mathcal{H}(n, M, r). Then \; \langle f, \lambda\rangle > r + \varepsilon \; for \; some \; \varepsilon > 0$; let F be a finite set with $|\lambda|(F) > 1 - \varepsilon/2M$(cf.(44)).

Let F be a neighborhood of f and W a neighborhood of , given by Claim A. Then

$$\mathcal{W} = \{v \in \Sigma: \langle f, \lambda\rangle > r + \varepsilon, |v|(W) > 1 - \varepsilon/2M\}$$

is a neighborhood of $\lambda \; in \; (\Sigma, w^*); cf. (4.5). Let \; (g, v) \in \mathcal{V} \times \mathcal{W}, \|g\| \leq M.$By Claim A. $g|w = f|w; hence \; |\langle f, v\rangle - (g, v)| \leq 2M. |v|(T, W) < 2M. \; \varepsilon/2M = \varepsilon$. Since $\langle f, v\rangle > r + \varepsilon$, we get$(g, v) > r, \; that \; is, (g, v) \in \mathcal{H}(n, M, r)$.

(C) We shall now use Lemma (2.2.9) for $K = \widehat{\mathbb{T}}$ to complete the proof of (46). Let us consider the unit ball (cf. (45))

$$\mathfrak{R} = \{f \in C_0(\widehat{\mathbb{T}}) : \|f\| \leq 1\}. \quad (54)$$

It is enough to check that the duality map $\langle , \rangle (\mathfrak{B}, w) \times (\Sigma, w^*) \to \mathbb{R}$ is Borel, that is that for any $r \in R$, the set

$\mathcal{A}(r) = \{(f, \lambda) \in \mathfrak{R} \times \Sigma : (f, \lambda) > r\}$is Borel in $(\mathfrak{B}, w) \times (\Sigma, w^*)$. (55)

Let us adapt the notation from Lemma(2.2.9) and consider

$$\mathfrak{R}_i = \mathfrak{R} \cap \mathfrak{D}_i, \qquad \Phi_i : \mathfrak{R}_i \to C_0(\widehat{\mathrm{T}}, D_i).$$

Let (cf. Lemma (2.3.9))

$$\mathcal{A}(r, i, k) = \{(f, \lambda) \in \mathfrak{R}_i \times \Sigma : \delta(i) < 1/k \ \ and \ \langle \Phi_i(f), \lambda \rangle > r + 1/k\}. (56)$$

Since the map Φ_i is continuous with respect to the weak topology, claim B yields that the sets $\mathcal{A}(r, i, k)$ are Borel in $(\mathfrak{R}_i, w) \times (\Sigma, w^*)$. Therefore, to prove (55), and hence to complete the proof of Proposition(2.2.8) , it is enough to show that

$$\mathcal{A}(r) = \bigcup_{i,k} \mathcal{A}(r, i, k). \quad (57)$$

Let $(f, \lambda) \in \mathcal{A}(r, i, k)$. Then $\langle \Phi_i(f), \lambda \rangle > r + 1/k$ (cf. (56)), and $\|f - \Phi_i(f)\| < \delta(i) < 1/k$, by Lemma (2.2.9). Since $\|\lambda\| = 1$, we get $\langle f, \lambda \rangle > r$, that is $(f, \lambda) \in \mathcal{A}(r)$;cf. (55). Conversely, let $(f, \lambda) \in \mathcal{A}(r)$, and let k be such that $\langle f, \lambda \rangle > r$ +2 / k. By Lemma (2.2.9), there is I with $f \in \mathfrak{D}_i$ and $\delta(i) < 1/k$. Then $f \in \mathfrak{R}_i$, $\|f - \Phi_i(f)\| < \delta(i) < 1/k$, and in effect $\langle \Phi_i(f), \lambda \rangle > r + 2\ /\ k\ - 1\ /\ k$, that is, $(f, \lambda) \in \mathcal{A}(r, i, k)$.

Let (cf. [74]) T be the set of all injective maps $t: \alpha \to N$ defined on countable ordinals such that $N \setminus t(\alpha)$ is infinite, equipped with the order $t \preccurlyeq t' \Leftrightarrow dom\, t \subseteq dom\, t'$, and t' extends t. Haydon [54] proved that there is a Choquet space S and a continuous injective function $\Phi: S \to (C(\hat{T}), \text{weak})$ with $\Phi(S)$ norm-discrete. Moreover, the space S has no isolated points and has weight 2^{N_0}, and by a theorem of Fremlin [76]. S contains a non-Borel set A(cf. also Lemma (2.2.6)). Then $\Phi(A)$ is not Borel in $(C(\hat{T}), \text{weak})$, being norm-discrete. On the other hand, Proposition (2.2.8), shows that the duality map is Borel measurable.

In the proof of Proposition (2.2.8), We have omitted some standard, but not quite trivial, details concerning Borel measurability of the maps in the proof.

(A) Let K be a compact space, and let M(K) be the space of Radon measures on K, endowed with the w^*-topology. Then , for any point $p \in K$, the function $\mu \longmapsto \mu(\{p\})$ is Borel measurable on M(K).

To begin, let us recall that for any compact $F \subseteq K$, the map $\mu \longmapsto |\mu|(F)$ (and hence $\mu \longmapsto |\mu|(K \setminus F)$) is Borel on M(K). In particular, the set $B = \{\mu \in M(K) : |\mu|(K) = 1\}$ is Borel. We claim that$C = \{\mu \in B : \mu(\{p\}) > 0\}$ is Borel in M(K). To that end, let S be the set of all pairs (a, b) of rational numbers with $0 < a < b \leq 1, b - a < a < a/2$. Then each set

$$B(a, b) = \{\mu \in B : \mu(\{p\}) \in (a, b)\}\text{Is Borel.}$$

and

$$\bigcup \{B(a, b) : (a, b) \in S\} = B \setminus \{\delta_{\{p\}}, -\delta_{\{p\}}\}.$$

Therefore, to check that C is Borel it is enough to make sure that:

$$C(a, b) = C \cap B(a, b)\text{is relatively open in } B(a, b), \text{ for } (a, b) \in S.$$

Let $\mu \in (a,b)$. Then $|\mu|(K \setminus \{p\}) = 1 - \mu(\{p\}) > 1 - b$, and hence there is a continuous $f$on K with $-1 \leq f \leq 1, p \notin$ supp f and $\int f d\mu > 1 - b$, Since $\mu(\{p\}) > a. and\ \mu \in C,\ \mu(\{p\}) > a$ and there is continuous g on K with $0 \leq$ g ≤ 1, sup g $\cap$ supp$f = \emptyset$ and $gd\mu > a$.
Let

$$V = \left\{v \in B(a,b): \int f d\mu > 1 - b, \int \mathrm{g} dv > a.\right\}$$

Then V is a neighborhood of μ in $B(a,b)$. Let us check that V $\subseteq C(a,b)$. If $v \in V$, then $\int f d\mu > 1 - b$ implies that $|v|(\text{supp g}) < b$,and since $|v|(\{p\}) > a$, we get $|v|(\text{supp g} \setminus \{p\}) < b - a < a/2$. On the other hand, $\int \mathrm{g}dv > a$, and hence $v(\{p\}) > a/2 > 0$, that is $v \in C$.

Now, $\{\mu: \mu(\{p\}) > 0\} = \{\mu: \mu/|\mu|(K) \in C\}$ is Borel, since the map $\mu \longmapsto \mu/|\mu|(K)$ is Borel; hence, for any $r > 0$, the set

$$\{\mu: \mu(\{p\}) > r\} = \{\mu: |\mu|(\{p\}) > r\} \cap \{\mu: \mu(\{p\}) > 0\}$$

is Borel, which ends a justification of (A).
(B) Now let us explain the Borel measurability of the map

$$(f,\lambda) \longmapsto f - f(\infty), \frac{\sigma(\lambda)}{\|\sigma(\lambda)\|} \qquad (*)$$

Used in the proof of Proposition (2.2.8) .
We appeal to the following.
Indeed, let $\{B_1, B_2, \cdots\}$ be a countable base in Z. Then an open set in$Y \times Z$ is of the form $U = \bigcup_i (V_i \times B_i)$, with each V_i open in Y , and therefore $w^{-1}(U) = \bigcup_i \big(u^{-1}(V_i) \cap v^{-1}(B_i)\big)$ is a Borel set.
Now, the remark implies that the map.

$$(f,\lambda) \longmapsto \big((f - f(\infty), \lambda), \lambda(\infty).\delta_\infty\big) = \big(f - f(\infty), (\lambda, \lambda(\infty).\delta_\infty)\big)$$

is Borel, and since $(\lambda, r\delta_\infty) \longmapsto \lambda - r\delta_\infty$ is continuous, the map.

$$(f,\lambda) \longmapsto \big((f - f(\infty), \lambda) - \lambda(\infty).\delta_\infty\big) = \big(f - f(\infty), \sigma(\lambda)\big)$$

is Borel.
The remark also yields the fact that $(f,\lambda) \longmapsto \big((f,\lambda), 1/\|\gamma\|\big) = \big(f, (\lambda, 1/\|\lambda\|)\big)$ is Borel, and since $(\lambda, r) \longmapsto r, \lambda$ is continuous, the map $(f,\lambda) \longmapsto \big((f,\lambda)/\|\gamma\|\big)$ is Borel .
In effect, the composition

$$(f,\lambda) \longmapsto \big(f - f(\infty), \sigma(\lambda)\big) \longmapsto \left(f - f(\infty), \frac{\sigma(\lambda)}{\|\sigma(\lambda)\|}\right),$$

that is, the map $(*)$, is Borel. Following Hansell [6], we call a Banach space E descriptive if there is a collection $\varepsilon = \bigcup_n \varepsilon_n$ of subsets of E, where each ε_n is a relatively discrete cover of $(\bigcup \varepsilon_n$, weak) and for any $x \in E\ and\ \ \varepsilon > 0\ there\ is\ A \in \varepsilon\ \ with\ x \in A$ and norm-diameter $(A) < \varepsilon$.

One can easily check that for any descriptive Banach space E, the duality map $\langle,\rangle: (E, \text{weak}) \times (E^*, \text{weak}^*) \to \mathbb{R}$ is Borel-measurable and Borel (E, weak)=Borel(E, norm); cf. [73], [53]. See [73], [74], [53], [56] for some important classes of compact spaces K whose functions spaces C(K) are descriptive, No example is known of a Banach space E with Borel (E, weak) = Borel (E, norm) that is not desriptive; cf. Oncina [88]. In all examples of function spaces C(K) with Borel (C(K), weak)≠Borel (C(K), norm) that we are aware of, there is a norm-discrete set in C(K) which is not Borel with respect to the weak topology. The ' discretization' argument used shows that this is always true for compact scattered spaces K, but we do not know if this is true in general.

Burke [73] addresses the problem of for which spaces X it is true that if Y is any space and $e: X \times Y \to \mathbb{R}$ is separately continuous, then e is Borel measurable. Theorem (2.2.1) shows that none of the infinite compact F-spaces X have this property. This partially answers [73] Another related result is the following observation. Let X be a Baire *p*-space (that is, all $G_\delta -$ sets in X are open) without isolated points, let Y be the space of real-valued continuous functions on X with the pointwise topology, and let $e: X \times Y \to \mathbb{R}$ be the evaluation map. Then e is not C-measurable (being separately continuous).

Kenderov, Kortezov and Moors [84] constructed a continuous map $\emptyset: E \to (\ell^\infty, \text{weak})$,defined on a compactly regularChoquet space which is not norm continuous at any point of E. The construction is based on some special games discussed. The approach used of yields a more direct construction, providing for any function space C(K) on an infinite compact F-space K such a map $\emptyset: E \to C_w(K)$, where E in addition has weight 2^{N_0}. (The weight of the domain of the map in [84] is greater than 2^{N_0}).

The reasoning can also be used to the following effect.

Proposition (2.2.10)[72]: Let K be a compact scattered space such that, for some $p \in K, K \setminus \{p\}$ has a continuous injection into a separable metrizable space.Then the evaluation map$\langle , \rangle: (C(K), \text{weak}) \times (\ell_1(K), \text{weak}^*) \to \mathbb{R}$ is Borel.

A variety of such compact spaces were constructed by van Douwen [91] and, under the continuum hypothesis. By Kunen (cf. [87]). In particular, for Kunen's space K (the continuum hypothesis is assumed), $C_w(K)$ is hereditarily Lindelof, has weight $\aleph_1$, and hence $\aleph_1$Borel sets. On the other hand, C(K) has a norm-discrete set of cardinality $\aleph_1$; hence there are 2^{N_1} norm-discrete sets in C(K) .

It follows that Borel (C(K), weak)≠Borel (C(K), norm).

Chapter 3
Problem of R.V. Kadison and Maximal Injective Subalgebras

We exhibit the first concrete examples of maximal injective von Neumann subalgebras in type II, factors. We solve two old problems of R. V. Kadison on the embeddings of the hyperfinite factor R.

Section (3.1): Maximal Abelian *-Subalgebras in Factors

For M be a factor von Neumann algebra and let $N \subset M$ be a subfactor. If M is the algebra of all bounded operators on a Hilbert space $\mathcal{H}$, $M = \mathfrak{B}(\mathcal{A})$, then by the well known theorem of von Neumann the bicommutant of N in M is equal to N. But if M is a continuous factor then in general the bicommutant of N in M is not equal to N (see [98]). Actually it seems that the typical and more interesting type of imbedding of N as a subfactor of a continuous factor M is such that the commutant of N in M is trivial, i.e. $N' \cap M = C$. For example, let $\boldsymbol{\alpha}: G \to$ Aut (N) be a properly outer action of the discrete group G on N. Then N is naturally imbedded in the crossed product algebra $N \times_{,} G$ and by the relative commutant theorem we have $N' \cap (N \times_\alpha G) = \mathbb{C}$.

A sufficient condition for a subfactor N of the factor M to have trivial relative commutant is that there exist an abelian *-subalgebra $A \subset N$ which is maximal abelian in M. (Indeed, because then $N' \cap M \subset A' \cap M = A \subset N$, so that $N' \cap M = \mathbb{C}$, N being a factor).

We show that under certain conditions Kadison's problem has an affirmative answer. The main result is the following. Let M be a separable factor and let $N \subset M$ be a semifinite subfactor such that $N' \cap M = \mathbb{C}$ and such that there exists a normal conditional expectation of M onto N.

Then there exists an abelian ,*-subalgebra A in N which is maximal abelian in M and which is semiregular in N (i.e. the normalizer of A in N generates a factor). In particular if M is a separable type II_1 factor then there exist normal conditional expectations onto all its weakly closed$*$ subalgebras, so that if N is a subfactor in M with $N' \cap M = \mathbb{C}$ then there exists a semiregular maximal abelian subalgebra of N which is also maximal abelian in M.
We also show by a counterexample that the hypothesis of separability is essential: if M^ω is the algebra defined as in [96], [101], for a non Γ type II_1 factor M and for a free ultrafilterω on $\mathbb{N}$, then by a theorem of A. Connes $M' \cap M^\omega = \mathbb{C}$, but no maximal abelian subalgebra of M is maximal abelian in M^ω.

We show some technical results concerning the algebraic condition $N' \cap M \subset N$, which is the natural generalization of the condition $N' \cap M = \mathbb{C}$ for the case when N is not a factor.

We mention some consequences of the main theorem and we give the counter example for the nonseparable case.

In what follows M will always denote a von Neumann algebra of countable type. All the subalgebras of M that we shall consider here will be selfadjoint, weakly closed and will contain the unit of M.

Let φ be a fixed normal faithful positive form on M. For $x \in M$ we denote by $\|x\|_\varphi = \varphi(x^*x)^{1/2}$ the Hilbert norm on M given by the scalar product $(x, y) \mapsto \varphi(y^* x)$. Denote by $\mathcal{H}_\varphi$the Hilbert space obtained by the completion of M in the norm $\| \quad \|_\varphi$.

Let $B \subset M$ be a subalgebra and $E: M \longrightarrow B$ a normal conditional expectation on B, with $\varphi o E = \varphi$. Then for $x \in M$ and $y \in B$ we have $\varphi(xy) = \varphi(E(xy)) = \varphi(E(x)y)$, so that $\varphi((x - E(x))y) = 0$. Thus $x - E(x)$ is orthogonal to B in the Hilbert spac $\mathcal{H}_\varphi$ and $E(x) \in B$ is the orthogonal projection of x on the closure$\overline{B}^\varphi$ of B in $\mathcal{H}_\varphi$" In particular it follows that the conditional expectation E is uniquely determined by the condition φ o E $= \varphi$.

Definition (3.1.1)[92]:A subalgebra $B \subset M$ is φ compatible with M if there exists a normal φ preserving conditional expectation of M onto B. By the above remarks this conditional expectation is unique and will be denoted by E_B^φ.

If $B \subset M$ is φ compatible with M then $E_B^\varphi(x) \in B$ is the orthogonal projection of x on $\overline{B}^\varphi \subset \mathcal{H}_\varphi$, so that if $B_2 \subset B_1 \subset M$ areφ compatible with M then $E_{B_2}^\varphi \circ E_{B_1}^\varphi = E_{B_2}^\varphi$ and $\left\|E_{B_2}^\varphi(x)\right\|_\varphi \leqq \left\|E_{B_1}^\varphi(x)\right\|_\varphi$for all $x \in m$. If in addition $A_1 \subset A_2 \subset B_2$ are also φ compatible with M, then

$$\left\|E_{B_2}^\varphi(x) - E_{A_2}^\varphi(x)(x)\right\|_\varphi \leqq \left\|E_{B_1}^\varphi(x) - E_{A_1}^\varphi(x)(x)\right\|_\varphi.$$

As usual $M^\varphi = \{y \in M|\varphi(xy) = \varphi(yx) \text{ for all } x \in M\}$ will denote the centralizer of φ in M. By the Pedersen-Takesaki theorem M^φ may be also characterized as the fixed point algebra of the modular automorphism group $\sigma_t^\varphi, t \in \mathbb{R}$, associated to φ(see [103], Chap. 2).

Recall that by the Takesaki theorem on the existence of conditional expectations, $B \subset M$ is φ compatible with M if and only if B is invariant under the modular automorphism group of M associated with φ, i.e. $\sigma_t^\varphi(B) = B, t \in \mathbb{R}$ (see [103], Chap. i0). Since $\sigma_t^\varphi(x) = x$ for $x \in M^\varphi$, $t \in \mathbb{R}$, it follows that any subalgebra $B \subset M^\varphi$ is φ compatible with M. Moreover if y e M commutes with B, then $\sigma_t^\varphi(y)$ commutes with $\sigma_t^\varphi(B) = B$ so that $\sigma_t^\varphi(B' \cap M) = B' \cap M, t \in \mathbb{R}$. Using again Takesaki's theorem it follows that $B' \cap M$ is φ compatible with M.

The following lemma gives a criterion for commutative subalgebras in M^φ r to be maximal abelian in M. The proof may be easily deduced from [94]. However we give here a complete proof for the sake of completeness.

Lemma (3.1.2)[92]: Let $\{A_i\}_{i \in I}$ be an increasing net of finite dimensional .-subalgebras in M^φ . Then $A = \overline{\bigcup_\iota A_\iota^w}$ is maximal abelian in M if and only if $\left\|E_{A_1' \cap M}^\varphi(X) - E_{A_t'}^\varphi(x)\right\|_\varphi \rightarrow 0$, for all $x \in M$.

Proof: Since $A \subset M^\varphi$ it follows that A and $A' \cap M$ are φ compatible with M. Note first that $\bigcap_i(A_i' \cap M) = (\bigcup_i A_i)' \cap M = A' \cap M$ and since $\{A_i\}_{i \in I}$, is increasing and $\{A_i' \cap M\}_{i \in I}$ is decreasing, it follows that the nets $\{E_{A_i}^\varphi(X)\}_{i \in I}$ and $\{E_{A_l' \cap M}^\varphi(x)\}_{i \in I}$ are Cauchy nets in $\mathcal{H}_\varphi$. We show that

(i) $\left\|E_{A_t}^\varphi(x) - E_A^\varphi(x)\right\|_\varphi \rightarrow 0, x \in M$.

(ii) $\left\|E_{A_t' \cap M}^\varphi(x) - E_{A' \cap M}^\varphi(x)\right\|_\varphi \rightarrow 0, x \in M$.

Since $E_{A_t}^\varphi(x)$ is the orthogonal projection of $E_A^\varphi(X)$on $A_i = \overline{A}_i^\varphi$, it follows that

$$\left\|E_A^\varphi(X) - E_{A_t}^\varphi(x)\right\|_\varphi = \inf\{\left\|E_A^\varphi(x) - y\right\|_\varphi |y \in A_i\}.$$

By the Kaplansky density theorem we get:

$$0 = \inf\{\|E_A^\varphi(x) - y\|_\varphi \Big| y \in \bigcup_i A_i\} = \inf_i \inf\{\|E_A^\varphi(x) - y\|_\varphi |y \in A_i\} = \|E_A^\varphi(X) - E_{A_t}^\varphi(x)\|_\varphi$$
$$= \lim_i \|E_A^\varphi(x) - E_{A_l}^\varphi(x)\|_\varphi,$$
and the proof of (i) is completed.

To prove (ii) remark that the net $\{E^\varphi_{A'_l \cap M}(x)\}_{i\in I}$ is bounded in the uniform norm, so that it has limit points in the weak topology on M. Moreover since the weakly closed algebras $A'_i \cap M$ decrease to $A' \cap M$, any weak limit point of this net is in $A' \cap M$. Let $y_o \in A' \cap M$ be a w-limit point of $\{E^\varphi_{A'_l \cap M}(x)\}_{i\in I}$ In particular it follows that the net $\{\varphi((x - y_0)^*(E^\varphi_{A'_l \cap M}(x) - y_0\}_{i\in I}$ has 0 as a limit point. Since $\varphi o E^\varphi_{A'_l \cap M} = \varphi)$ we obtain that

$$\varphi(E^\varphi_{A'_t \cap M}((x - y_0)^* (E^\varphi_{A'_l \cap M}(x) - y_0)))$$
$$= \varphi(E^\varphi_{A'_t \cap M}((x - y_0)^* (E^\varphi_{A'_l \cap M}(x) - y_0)) = \left\|E^\varphi_{A'_l \cap M}(x) - y_0\right\|^2_\varphi$$

has 0 as a limit point and since it is Cauchy it follows that $\left\|E^\varphi_{A'_l \cap M}(x) - y_0\right\|_\varphi \to 0$ and finally

$$\left\|E^\varphi_{A'_l \cap M}(x) - E^\varphi_{A' \cap M}(x)\right\| I \leqq \left\|E^\varphi_{A'_l \cap M}(x) - y_0\right\|_\varphi \to 0$$

The statement follows now easily since $A \subset M$ is maximal abelian if and only if $A' \cap M = A$, or equivalently $E_A^\varphi = E^\varphi_{A' \cap M}$.

We point out that by the normality of the conditional expectations $E^\varphi_{A' \cap M}$ and E^φ_A, if $E^\varphi_{A' \cap M}(x) = E^\varphi_A(X)$ for all x in a total subset of M then $E^\varphi_{A' \cap M} = E^\varphi_A$, so that to decide that A is maximal abelian in M it is enough to have $\left\|E^\varphi_{A'_l \cap M}(x) - E^\varphi_{A_1}(X)\right\|_\varphi \to 0$, for x in a total subset of M.

Remark that for $A_o = M^\varphi$, finite dimensional and abelian, $E^\varphi_{A_o}, E^\varphi_{A'_o \cap M}$ are given in the following way: if $e_1, e_2, \ldots, e_n, \in A_o$ are the minimal projections of A_0 and $x \in M$, then

$$E^\varphi_{A_o}(x) = \sum_{i=1}^n \frac{\varphi(e_i x e_i)}{\varphi(e_i)} e_i$$
$$E^\varphi_{A'_o \cap M} x) = \sum_{i=1}^n e_i x e_i.$$

Note also that any abelian von Neumann algebra A may be obtained as the closure of an increasing net of finite dimensional ,*subalgebras. If A is separable then A is single generated and it can be obtained as the closure of an increasing sequence of finite dimensional abelian ,-subalgebras.

Let *M* be a yon Neumann algebra of countable type and let cp be a fixed normal faithful positive form on M. We shall prove some characterizations of the property $N' \cap M \subset N$, in the case $N \subset M^\varphi$. Since the condition $N' \cap M \subset N$ is equivalent to $N' \cap M = \mathfrak{Z}(N)$, the center of N, it follows that if N is a factor and $N' \cap M \subset N$ then M is a factor and $N' \cap M = \mathbb{C}$.

We point out that M^{φ} is a finite algebra, since the restriction of φ to M^{φ} is a faithful trace. Thus $N \subset M^{\varphi}$ is necessary finite.

First we show that the condition $N' \cap M \subset N$ can be localized by reduction with projections in N. Recall that if e is a projection of the von Neumann algebra $M \subset \mathfrak{B}(\mathcal{H}$, then the reduced von Neumann algebra of M with respect to e is the algebra $M_e \overset{\text{def}}{=} eMe \subset \mathfrak{B}(e\mathcal{H})$. The commutant of M_e in $\mathfrak{B}(e\mathcal{H})$ is $M' e \subset \mathfrak{B}(e\mathcal{H})$.

Lemma (3.1.3)[92]: If $N \subset M$ and $e \in N$ is a projection, then $(N_e)' \cap M_e = (N' \cap M)_e$. In particular if N and M are factors and $N' \cap M = \mathbb{C}$, then $(N_e)' \complement \cap M_e = \mathbb{C}$.

Proof. The inclusion $(N' \cap M)_e \subset (N_e)'c \cap M_e$ is trivial. For the opposite inclusion le $z(e)$ be the central support of e in N and let $x \in (N_e)' \cap M_e$. Since $(N_e)' = N'e$, there exists $x' \in N'$ such that $x'e = ex'e = x \in eMe$. We show that $xz(e) \in M$.

Let $f \in N$ be a projection, maximal with the properties that $e \leqq f \leqq z(e)$ and $x'f \in M$. If $z(e) - f \neq 0$, then $(z(e) - f)Ne \neq 0$ and it follows that there exists a partial isometry $v \in N$ such that $v * v \leqq e$ and $vv * \leqq z(e) - f$ We have $M \ni v(ex'e)\, v *= vx'v^* = x'vv^*$, so that $x'(vv^* + f) \in M$, which contradicts the maximality of f It follows that $f = z(e)$, so that $x'z(e) \in M$ and $x = ex'e = e(x'\, z(e))e \in e(N' \cap M)e$.

Lemma (3.1.4)[92]: Let $N \subset M^{\varphi}$ be such that $N' \cap M \subset N$. Given $\varepsilon > 0$ and $x \in M, x \neq 0, x$ φ-orthogonal to N (i.e. such that $E_N^{\varphi}(x) = 0$), there exists a unitary $u \in N$ such that

$$\|uxu^* - x\|_{\varphi}^2 > (2 - \varepsilon)\, \|x\|_{\varphi}^2,$$

Proof : Denote by $K_x = \overline{co}^w \{vxv^* | v \text{ unitary in } N\}$. First we show that $0 \in K_x$.

Since K_x is a weakly compact convex subset of M, by the inferior semicontinuity of the norm $\| \quad \|_{\varphi}$ it follows that there exists an element $y_0 \in K_x$ such that

$$\|y_0\|_{\varphi} = \inf\{\|y\|_{\varphi} | y \in K_x\}.$$

Since $\| \quad \|_{\varphi}$ is a Hilbert norm and K_x is convex, it follows that y_0 is the unique element in K_x with this property. But $v\, K_x v^* \subset K_x$ for all unitaries v in N. In particular $vy_0v^*eK_x$. Since $v \in M^{\varphi}$ we have $\|vy_0v^*\|_{\varphi} = \|y_0\|_{\varphi}$, so that $v\, y_0 v^* = y_0$ for all unitaries v in N. Consequently $y_0 \in N' \cap M \subset N$. By the hypothesis, $N \subset M^{\varphi}$ and x is orthogonal to N, so that $\varphi(vxv^*y) = \varphi(xv^*\, y\, v) = 0$ for all $y, v \in N$. It follows that all elements in K_x are orthogonal to N. Thus $y_0 \in N$ and y_0 is orthogonal to N, that is $y_0 = 0$.

Suppose now that $\|vxv^* - x\|_{\varphi}^2 \leqq (2 - \varepsilon)\|x\|_{\varphi}^2$ for all unitaries v in N. We obtain that $\|vxv^*\|_{\varphi}^2 + \|x\|_{\varphi}^2 - 2\,\mathrm{Re}\varphi(x^*vxv^*) \leqq (2 - \varepsilon)\|x\|_{\varphi}^2$ so that $2\,\mathrm{Re}\varphi(x^*vxv^*) \geqq \varepsilon\|x\|_{\varphi}^2$ for all unitaries v in N. Thus we get $2\mathrm{Re}\varphi(x^*y) \geqq \varepsilon, \|x\|_{\varphi}^2$ for every $y \in K_x$ and in particular $0 \geqq \varepsilon\|x\|_{\varphi}^2$, which is a contradiction.

Lemma (3.1.5)[92]: Let $N \subset M^{\varphi}$ be such that $N' \cap M \subset N$. If $\varepsilon > 0$ and $x_1 \ldots\ldots x_n \in M, x_i \neq 0$, are φ-orthogonal to N, then there exists a finite dimensional abelian ∗ -subalgebra $A_0 \subset N$ such that

$$\left\| E_{A_0' \cap M}^{\varphi}(x_i) \right\|_{\varphi} < \varepsilon\, \|x_i\|_{\varphi}\ 1 \leqq i \leqq n.$$

Proof: Consider first one element $x \in M, x$ orthogonal to N. By the preceding lemma there exists a unitary $u \in N$ such that $\|uxu^* - x\|_{\varphi} > \|x\|_{\varphi}$. Choose $e_1, e_2, \ldots, e_s \in N$ to be spectral projections of u, such that $e_1 + e_2 + \ldots + e_s = 1$ and such that for suitable scalars $\lambda_1, \ldots, \lambda_2, |\lambda_i| = 1$, we have $\|\sum_{i=1}^{s} \lambda_i e_i - u\|$small enough to ensure that

$$\left\|\left(\sum_{i=1}^{s}\lambda_i e_i\right)x\left(\left\|\sum_{j=1}^{s}\bar{\lambda}_j e_j - u\right\|\right) - x\right\|_{\varphi} \geqq \|x\|_{\varphi}$$

Since $e_i \in M^{\varphi}$, the elements $\{e_i x e_j\}_{1\leqq i,j\leqq s}$ are orthogonal in $\mathcal{H}_{\varphi}$ and by the inequalities $2 \geqq |\lambda_i\bar{\lambda}_j - 1|$, we get:

$$4\|x\|_{\varphi}^2 - 4\left\|\sum_i e_i x e_i\right\|_{\varphi}^2 = 4\left\|\sum_{i\neq j} e_i x e_j\right\|_{\varphi}^2$$

$$\geqq \left\|\sum_{i\neq j}(\lambda_i\bar{\lambda}_j - 1)e_i x e_j\right\|_{\varphi}^2 = \left\|\left(\sum_{i=1}^{s}\lambda_i e_i\right)x\left(\sum_{j=1}^{s}\bar{\lambda}_j e_j\right) - x\right\|_{\varphi} \geqq \|x\|_{\varphi}^2$$

From the first and the last terms of the inequalities we get:

$$\left\|\sum_i e_i x e_i\right\|_{\varphi}^2 \leqq (3/4)\|x\|_{\varphi}^2$$

For the general case, suppose that $m \geqq l$ is such that $(3/4)^m < \varepsilon^2$. Let $e_1, \ldots, e_s \in N$ be mutually orthogonal projections and $l \leqq k \leqq n, p < m$ be such that:

$$\left\|\sum_{i=1}^{s} e_i x_j e_i\right\|_{\varphi}^2 \leqq (3/4)^m \left\|x_j\right\|_{\varphi}^2, \text{for } j < k,$$

$$\left\|\sum_{i=1}^{s} e_i x_k e_i\right\|_{\varphi}^2 \leqq (3/4)^p \|x_k\|_{\varphi}^2$$

Applying Lemma (3.1.3) ,and the first part of the proof for each pair of algebras $N_{e_i} \subset (M_{e_i})^{\varphi_{e_i}}$, and the element $e_i x_k e_i \in e_i M e_i$, which is φ_{e_i}, orthogonal to $e_i N_{e_i}$ we get a set of pairwise orthogonal projections $f_1, \ldots, f_t, \in N$, refining $e_i, \ldots, e_s$ and such that:

$$\left\|\sum_{i=1}^{t} f_i x_k f_i\right\|_{\varphi}^2 \leqq (3/4)\sum_{i=1}^{s}\|e_i x_k e_i\|_{\varphi}^2$$

$$= (3/4)\sum_{i=1}^{s}\|e_i x_k e_i\|_{\varphi}^2 \leqq (3/4)^{p+1}\|x_k\|_{\varphi}^2$$

Since each e_i is the sum of some f_i we also have for $j < k$:

$$\left\|\sum_{i=1}^{t} f_i x_j f_i\right\|_{\varphi}^2 \leqq \left\|\sum_{i=1}^{s} e_i x_j e_i\right\|_{\varphi}^2 \leqq (3/4)^m \left\|x_j\right\|_{\varphi}^2$$

By induction, we get a finite set of projections $g_1, \ldots, g_t$ in N, mutually orthogonal, such that:

$$\left\|\sum_{i=1}^{i} g_i x_j g_i\right\|_{\varphi}^{2} \leqq (3/4)^m \|x_j\|_{\varphi}^2 < \varepsilon^2 \|x_j\|_{\varphi}^2, \text{for all } l \leqq j \leqq n.$$

Taking A_0 to be the algebra generated by $\{g_i\}_{1\leqq i\leqq l}$ the statement follows.

Theorem(3.1.6)[92]: Let M be a von Neumann algebra of countable type with a faithful normal positive form φ. Let $N \subset M^{\varphi}$ be such that $N' \cap M \subset N$. If $x_1 \ldots, x_n \in M$ and $\varepsilon > 0$, then there exists a finite dimensional abelian *-subalgebra $A \subset N$, such that

$$\left\|E^{\varphi}_{A' \cap M}(x_i) - E^{\varphi}_{A}(x_i)\right\|_{\varphi} < \varepsilon, i = 1, \ldots, n.$$

Moreover, if M and N are factors and $N' \cap M = \mathbb{C}$, then A can be chosen such that its minimal projections are equivalent in N.

Proof: Denote by $\varepsilon' = \varepsilon\left(1 + \sum_{i=1}^{n} \|x_i\|_{\varphi}\right)^{-1}$. Let $x_i' = E_N^{\varphi}(x_i), x_i'' = x_i - x_i', i = 1, .., n$.
By Lemma (3.1.5) we get a finite dimensional, abelian $*$-subalgebra AocN such that

$$\left\|E^{\varphi}_{A_0' \cap M}(x_i'')\right\|_{\varphi} \leqq (\varepsilon'/2) \|x_i''\|_{\varphi}, i = 1, \ldots, n.$$

Let $\{e_i\}_i \subset A_0$ be the minimal projections of A_0.
Consider the algebra N_{e_i}, and take a maximal abelian ,-subalgebra in N_{e_i} (there is one by Zorn's lemma). Apply Lemma (3.1.2) for N_{e_i} and this maximal abelian subalgebra to obtain a finite dimensional abelian ,*subalgebra $A_i \subset N_{e_i}$, such that:

$$\left\|E^{\varphi}_{A_0' \cap M}(e_i x_j' e_i) - E^{\varphi_{e_i}}_{A_l}(e_i x_j' e_i)\right\|_{\varphi_{e_i}} \leqq (\varepsilon'/2) \left\|e_i x_j' e_i\right\|_{\varphi}, j = 1, \ldots, n.$$

If we let $A = \sum_i A_i$, then $A \subset N$. Since x_j') are also in N we get:

$$\left\|E^{\varphi}_{A' \cap M}(x_j') - E^{\varphi}_{A}(x_j')\right\|_{\varphi}^{2} =, \left\|E^{\varphi}_{A' \cap N}(x_j') - E^{\varphi}_{A}(x_j')\right\|_{\varphi}^{2}$$
$$= \sum_{\mathbf{i}} \left\|E^{\varphi_{e_i}}_{A_l \cap N_{e_i}}(e_i x_j' e_i) - E^{\varphi_{e_i}}_{A_l}(e_i x_j' e_i)\right\|_{\varphi_{e_i}}^{2}$$
$$\leqq (\varepsilon'^2/4) \sum_{\mathbf{i}} \left\|e_i x_j' e_i\right\|_{\varphi_{e_i}}^{2} \leqq (\varepsilon'^2/4) \left\|x_j'\right\|_{\varphi}^{2}$$

Finally we get:

$$\left\|E^{\varphi}_{A' \cap M}(x_j) - E^{\varphi}_{A}(x_j)\right\|_{\varphi}$$
$$\leqq \left\|E^{\varphi}_{A' \cap M}(x_j) - E^{\varphi}_{A}(x_j)\right\|_{\varphi} + \left\|E^{\varphi}_{A' \cap M}(x_j'') - E^{\varphi}_{A}(x_j'')\right\|_{\varphi}$$
$$\leqq (\varepsilon'/2) \left\|x_j'\right\|_{\varphi} + \left\|E^{\varphi}_{A_0' \cap M}(x_j'') - E^{\varphi}_{A_0}(x_j'')\right\|_{\varphi}$$
$$\leqq (\varepsilon'/2) \left\|x_j'\right\|_{\varphi} + \left\|x_j''\right\|_{\varphi}) \leqq \varepsilon' \left\|x_j\right\|_{\varphi} < \varepsilon, \ 1 \leqq j \leqq n.$$

It is not hard to see that if N is a factor then we can modify a such that its minimal projections have rational trace in N and such that the above inequalities still hold.
Taking an appropriate refinement of A the last part of the statement follows easily.

For the results M will allways denote a separable von Neumann algebra (i.e. with separable predual).
The next definition was first introduced by J. Dixmier (see [97]).

Definition(3.1.7)[92]: Let M be a factor and let $A \subset M$ be a maximal abelian *-subalgebra of M. Denote by $.\mathcal{N}(A) = \{u \in M | u \text{ unitary}, uAu^* = A\}$ the normalize of A in M and by $\mathcal{N}(A)$ the weakly closed .-subalgebra generated by $\mathcal{N}(A)$ in M (in fact $N(A) = \overline{\text{span}}^w \mathcal{N}(A)$). If N(A)=M then A is called regular. If N(A) is a subfactor of M then A is called semiregular.

Theorem(3.1.8)[92]: Let M be a separable factor and let $N \subset M$ be a semifinite subfactor of M. Suppose that $N' \cap M = \mathbb{C}$ and that there exist a normal conditional expectation of M onto N. Then there exists a maximal abelian *-subalgebra A in N which is maximal abelian in M and which is semiregular in N.

If in addition N is hyperfinite then A may be chosen to be regular in N.

Proof: Let $E: M \to N$ be the normal conditional expectation. By a result of A. Connes, E is unique and faithful (see [103], Prop. 10.17).

Note that if N is a type I factor and $N' \cap M = \mathbb{C}$, then N=M and the statement is trivial. Thus, we have to prove only the case N is of type II.Suppose N is of type II_1 and let $\{x_n\}_{n \geqq l}$ be a total sequence in M. If τ isthe trace on N, then let $\varphi = \tau \circ E$. Since τ and E are faithful, φ is faithful, and clearly $N \subset M^{\varphi}$. Using Theorem (3.1.6) we construct recursively an increasing sequence of matrix subalgebras $\{N_n\}_{n \geqq l}$ of N, each of them with a set of matrix units $\{e_{ij}^n\}_{1 \leqq i, j \leqq k_n}$ such that:

(i) $\sum_{i=1}^{k_n} e_{ii}^n = 1,$

(ii) every e_{ij}^p, for $p \leqq n$, is the sum of some e_{kl}^n,

(iii) if A_n denotes the diagonal algebra generated by $\{e_{ii}^n\}_{1 \leqq i \leqq k_n}$ then

$$\left\| E^{\varphi}_{A_n' \cap M}(x_j) - E^{\varphi}_{A_n}(x_j) \right\|_{\varphi} \leqq 2^{-n}, 1 \leqq j \leqq n.$$

Suppose this construction is done for $1 \leqq n \leqq m$.

Let $e = e_{11}^m \in A_m \subset N_m \subset N$ be a minimal projection in A_m. Since $(N_e)' \cap M_e = \mathbb{C}$, we can apply Theorem (3.1.6) to obtain a finite dimensional abelian ,*-subalgebra A_0 in N_e such that

$$\left\| E^{\varphi_e}_{A_n' \cap M_e}(e_{li}^m x_k e_{il}^m) - E^{\varphi_e}_{A_0}(e_{li}^m x_k e_{il}^m) \right\|_{\varphi_e} \leqq k_m^{-1} 2^{-(m+1)}$$

$$k = 1, 2, \ldots, m + 1, i = 1, 2 \ldots, k_m .$$

Suppose all the minimal projections of A_0 are equivalent and let $\{e'_{sk}\}_{1 <= s, k \leqq p}$ be matrix units in N_e(which is a factor) such that $\{e'_{kk}\}_{1 \leqq k \leqq p}$ generates A_0. Take $N_{m+1} \subset N$ to be the matrix algebra generated by the matrix units

$$\{e_{ij}^{m+1}\}_{1 \leqq i, j \leqq k_{m+1}} \stackrel{\text{def}}{\equiv} \{e'_{sk} e_{ij}^m\}_{\substack{1 \leqq j, j \leqq k_m \\ 1 \leqq s, k \leqq p}},$$

as is easily seen we have

$$\text{span}\{e_{jj}^{m+1}\}_{1 \leqq j \leqq k_{m+1}} = A_{m+1} = \sum_{i=1}^{k_m} e_{il}^m A_0 e_{li}^m,$$

and for $x \in M$ we have

$$\left\|E^{\varphi}_{A'_{m+1}\cap M}(x) - E^{\varphi}_{A_{m+1}}(x)\right\|^2_{\varphi} =. \sum_{i=1}^{k_m} \left\|E^{\varphi_e}_{A'_0\cap M_e}(e^m_{1i}x_k e^m_{il}) - E^{\varphi_e}_{A_0}(e^m_{li}x_k e^m_{il})\right\|^2_{\varphi_e}$$

By the inequalities (*) we get:

$$\left\|E^{\varphi}_{A'_{m+1}\cap M}(x_j) - E^{\varphi}_{A_{m+1}}(x_j)\right\|_{\varphi} \leqq k_m . k_m^{-1} . 2^{-(m+1)} = 2^{-(m+1)}, j = 1, \ldots, m+1 .$$

Thus, if $A = \overline{\cup_{n\geqq 1} A^w_n}$, $R = \overline{\cup_{n\geqq 1} N^w_n}$, then $R \subset N$, A is a maximal abelian ,*sub-algebra in M by Lemma(3.1.2) and clearly A is regular in R.

Consequently, if N(A) denotes the algebra generated by the normalizer of A in N, then $A \subset R \subset N(A)$, so that the center of N(A) is included in R. It follows that N(A) is a factor, since R is a factor.

Suppose now that in addition N is hyper finite and let $\{Y_n\}_{n\geqq 1}$ be a total sequence in N. We construct recursively an increasing sequence of matrix subalgebras $\{N_n\}_{n\geq 1}$ in N, with matrix units $\{e^n_{ij}\}_{1\leqq i,\leqq k_n}$ such that the preceding conditions (i), (ii), (iii) hold together with the condition

(iv) $\left\|y_i - E^{\varphi}_{N_n'}(y_j)\right\|_{\varphi} \leqq 2^{-n}, 1 \leqq j \leqq n.$

Suppose this construction is done for $1 \leqq n \leqq m$. As before, we can get a matrix algebra N^0_{m+1} such that $N_m \subset N^0_{m+1} \subset N$ and such that conditions (i), (ii), (iii) are fulfilled. Since N is hyperfinite we can take a matrix subalgebra N_{m+1} in N, $N_{m+1} \supset N^0_{m+1}$,) also holds.

If $A = \overline{\cup_n A^w_n}$ and $R = \overline{\cup_n N^w_n}$, then A is regular in R and by condition iv), $R = N$.

Suppose now that N is of type II_∞. Let $e \in N$ be a finite projection of N. If $\mathcal{H}$ is a separable infinite dimensional Hilbert space, then M is isomorphic to $M_e \overline{\oplus} \mathfrak{B}(\mathcal{H})$and the inclusion of N in M becomes the inclusion $N_e \overline{\oplus} \mathfrak{B}(\mathcal{H}) \subset M_e \overline{\oplus} \mathfrak{B}(\mathcal{H})$ By Lemma(3.1.3) $(N_e)' \cap M_e = \mathbb{C}$ and the restriction of E to M_e gives a normal conditional expectation from $M_e$onto N_e. By the first part of the proof it follows that there exists an abelian *-subalgebra A_1 in $N_e$which is maximal abelian in M e and whose normalizer in N e generates a subfactor $N(A_1) \subset N_e$,. Let $A_2 \subset \mathfrak{B}(\mathcal{H})$ be an atomic maximal abelian subalgebra_in $\mathfrak{B}(\mathcal{H})$ Then A_2 is regular in $\mathfrak{B}(\mathcal{H})$. It follows that $A = A_1 \overline{\oplus} A_2 \subset N_e \overline{\oplus} \mathfrak{B}(\mathcal{H})$ is maximal abelian in $M_e \overline{\oplus} \mathfrak{B}(\mathcal{H})$ and the normalize of A in $N_e \overline{\oplus} \mathfrak{B}(\mathcal{H})$ generates a factor (this is in fact $N(A_1 \overline{\oplus} \mathfrak{B}(\mathcal{H}))$). If N is hyperfinite then by Connes' theorem N_e is the hyper finite II_1 factor and applying again the first part of the proof a may be obtained to be regular in N.

Theorem(3.1.9)[92]: Let M be a separable von Neumann algebra and let $N \subset M$ be a semifinite von Neumann subalgebra of M. If $N' \cap M \subset N$ and if there exists a normal conditional expectation of M onto N then there exists a maximal abelian * -subalgebra in N which is maximal abelian in M.

Proof: Since $N' \cap M \subset N$, it follows that the normal conditional expectation of M onto N is unique (see [103], Prop. 10.17). Let $\{e_n\}_{n\geqq 1}$ be a sequence of finite projections in N such that $\sum_n e_n = 1$. For each pair of algebras $N_{e_k} \subset M_{e_k}, k \geqq 1$, using Theorem(3.1.6) and arguing as in the first part of the proof of Theorem (3.1.8) it follows that there exist a maximal abelian .-subalgebra A_k of M_{e_k}, contained in N_{e_k}. Then $A = \sum A_k$ is a subalgebra of N and it is maximal abelian in M.

Examples(3.1.10)[92]: (i) By Takesaki's theorem a sufficient condition for the existence of a normal conditional expectation of M onto a subalgebra N, is that N is in the centralizer of some normal faithful state on M (note that in this case N is necessary finite). In particular if M is a type II_1 factor then there exist normal conditional expectations onto all its von Neumann subalgebras. Thus, if M is separable $N \subset M$ and $N' \cap M = \mathbb{C}$, then by Theorem (3.1.8) there exists a semiregular maximal abelian subalgebra in N which is maximal abelian in M.

(ii) Let N be a separable semifinite von Neumann algebra and let $\alpha: G \to Aut(N)$ be a properly outer action of the countable discrete group G. By the relative commutant theorem we have $N' \cap (N \times_\alpha G) \subset N$ (see [103], Chap. 22) and since there is a normal conditional expectation of $N \times_\alpha G$ onto N, Theorem (3.1.9) applies, so that there exists a maximal abelian subalgebra A in N which is maximal abelian in $N \times_\alpha G$. Moreover if N is a factor then A may be chosen to be semiregular in N.

Proposition (3.1.11)[92]: Let M be a separable type II factor and let $A \subset M$ be amaximal abelian *-subalgebra generated by finite projections. If A is semiregular then A is contained in some hyperfinite subfactor of M, in which it is regular.

Proof: Let $.\mathcal{N}(A)$ *be* the normalizer of A in M and N(A) the weakly closed subalgebra generated by $\mathcal{N}(A)$.

Consider the groupoid $\mathcal{g} = \{v \in M | v$ partial isometry of the form $v = up, u \in \mathcal{N}(A), p \in A\}$. First we show that given two finite equivalent projections, $e, f \in A, ef = 0,$ there exists a partial isometry, $v \in \mathcal{g}$, such that $v^* v = e, vv^* = f$.

Denote by $\wp$ the family of pairs (p, w), where $p \in A, p \leqq e$, and $w \in \mathcal{g}$, such that $w^*w = p$ and $ww^* \leqq f$ Define a partial order in $\wp$ by $(P_0, W_o) < (P_1, w_1)$, if $P_o \leqq P_1, P_o \neq P_1, W_o = w_1 P_o$. By Zorn's lemma we obtain a maximal totally ordered family in $\wp$. Such a family has a countable cofinal subfamily so that it clearly has a maximal element (p', v'). Suppose $p' \neq e$ and denote $p" = e - p', q" = f - v'v' *$. (Note that $p"$ is equivalent with $q"$ in M.)

If $p"u^*q"u = O$ for any unitary $u \in \mathcal{N}(A)$ then the projection $g = \vee\{u^*q"u | u \in. \mathcal{N}(A)\} \in A$ satisfies $p"g = 0$ and $* gu = g$ for all u $\epsilon \mathcal{N}(A)$. It follows that g is a nonscalar element in A which commutes with $\mathcal{N}(A)$and thus it commutes with $N(A) = \overline{span}^w \mathcal{N}(A)$. This is a contradiction, since $\mathcal{N}(A))$ is a factor.

If$p"u^*q"u = P_o \neq 0$ for some $u \in \mathcal{N}(A)$ then $Po \leqq P", up_o$ is in the groupoid $\mathcal{g}$ and $v' + uP_0$ is also in $\mathcal{g}$. Thus $\wp \ni (p' + p_0\, v' + up_0) > (p', v')$, which is in contradiction with the maximality of (p', v'). Thus $p' = v'^*\, v' = e, v'\, v'^* = f, v' \in \mathcal{g}$.

We prove now the proposition in the case M is of type II_1 .

Let τ be the trace on M and let $\{a_n\}_{n \geqq 1} \subset A$ be a total sequence in A. We construct by induction an increasing sequence of matrix subalgebras $\{M_n\}_{n \geqq 1}$ of M, each of them with a set of matrix units $\{e^n_{ij}\}_{1 \leqq i,j \leqq k_n}$, such that:

(a)$\sum_{i=1}^{k_n} e^n_{ii} = 1$

(b) e^n_{ii} are in the groupoid $\mathcal{g}$, for all n, i, j,

(c) every e^p_{kl} , for $p \leqq n$, is the sum of some e^n_{ij},

(d) if A. denotes the diagonal algebra generated by $\{e^n_{ij}\}_{1 \leqq i \leqq k_n}.$, then $A_n \subset A$

and

$$\left\| a_j - E^t_{A_n}(a_j) \right\|_t \leqq 2^{-n}, \quad 1 \leqq j \leqq n.$$

Suppose this construction is done for $n \leqq m$.
Consider the elements $\{e^{m}_{1i}\, a_j\, e^{m}_{i1}\}_{\substack{1\leqq i\leqq k_m \\ 1\leqq j\leqq m+1}}$ which are in $e^{m}_{11}Ae^{m}_{11}$ (since $e^{m}_{1i} \in$ $\mathcal{G}$). Let $\{e'_{pp}\}_{1\leq p\leq s}$ be equivalent projections in $e^{m}_{11}Ae^{m}_{11}\sum_{p=1}^{s} e'_{pp} = e^{m}_{11}$ such that if $A_0 = span\ \{e'_{pp}\}_{1\leq p\leq s}$ then

$$\left\|e^{m}_{1i}a_je^{m}_{i1} - E^{t}_{A_0}\left(e^{m}_{1i}a_je^{m}_{i1}\right)\right\|_t \leqq k^{-1}_m 2^{-(m+1)}, 1 \leqq i \leqq k_m, 1 \leqq j \leqq m+1.$$

By the first part of the proof it follows that we can complete the set $\{e'_{pp}\}_p$ to a set of matrix units $\{e'_{pr}\}_{1\leqq p,r\leqq s}$ in the groupoid $\wp$.

If $\{e^{m+1}_{ij}\}_{1\leqq i,j\leqq k_{m+1}} \overset{\text{def}}{=} \{e'_{pr}e^{m}_{ij}\}_{\substack{1\leqq p,r\leqq s \\ 1\leqq i,j\leqq k_m}}$ and M_{m+1} is the algebra generated by $\{e^{m+1}_{ij}\}_{1\leqq i,j\leqq k_{m+1}}$ then conditions (i)-(iv) are clearly fulfilled.
If $B = \overline{\bigcup_{n\geqq 1} A^{w}_{n}}$ and $R = \overline{\bigcup_{n\geqq 1} M^{w}_{n}}$ then R is hyperfinite, B is regular in R and by condition (iv), B =A.
In the case M is a type II_∞ factor, there exists a sequence of finite projections $\{e_n\}_{n\geqq 1}$. in A, which are equivalent in M, such that $\sum_{n\geqq 1} e_n = 1$ (this is because A is generated by finite projections and because it has no minimal projections). Now we can apply the first part of the proof for pairs of projections e_n, e_{n+1}, and the type II_1 case, to conclude also the type II_∞ case.
Corollary (3.1.12)[92]: Every separable type ll factor M contains a hyperfinite subfactor R such that $R' \cap M = \mathbb{C}$.
Corollary (3.1.13)[92]: Every separable type II factor M has a maximal abelian *-subalgebra which is regular in some hyperfinite subfactor of M and thus, in particular, it is semiregular in M.

We mention that the existence of semiregular maximal abelian ,*-subalgebras in factors was recently shown to be important in connection with the Stone-Weierstrass theorems for C*-algebras, in [93].

We construct a counterexample for the nonseparable case. Let M be a separable type II_1 factor with finite trace τ and ω a free ultrafilter on N. Denote by M^ω the quotient of the von Neumann algebra $I^\infty(\mathbb{N}, M)$ by the zero ideal of the trace τ_ω. defined by $\tau_\omega((x_n)_n) = \lim_{k\to\omega} \tau(x_k)$ (see [96], [101]). Then M^ω is a finite factor and M is canonically imbedded in M^ω. By a well known result of A. Connes ([95]) if M has not the property Γ of Murray and von Neumann, then $M' \cap M^\omega = \mathbb{C}$. But if $A \subset M$ is any abelian .-subalgebra in M then A is far from being maximal abelian in M^ω. This follows easily by a direct argument. It is also a consequence of the following:
Proposition(3.1.14)[92]: If M is a type II_1 factor then no maximal abelian *-subalgebra of M^ω is separable.

Proof: Denote by $\pi_\omega$the natural projection of $l^\infty(\mathbb{N}, M))$ onto M^ω Let $B \subset M^\omega$ be a maximal abelian *-subalgebra and suppose B is separable. Then B is generated as a weakly closed .-subalgebra by a positive element $a \in B$. Let $(a_n)_{n\geqq 1}$ be a sequence of positive elements in M, with $\pi_\omega((a_n)_{n\geqq 1}) = a$ Take $A_n \subset M$ to be a maximal abelian .subalgebra in M such that $a_n \in A_n$ Denote by$\tilde{B}$the subalgebra of all sequences $(b_n)_{n\geqq 1}$ in $l^\infty(\mathbb{N}, M)$with $b_n \in A_n$. Then $\pi_\omega\ (\tilde{B})$ is commutative and $\pi_\omega\ \left(\tilde{B}\right) \supset B\ so\ that\ \pi_\omega\ (\tilde{B}) = B$.

Since M^{ω} is a continuous factor and B is maximal abelian and separable, one can find projections $\{e_{k,n}\}_{\substack{2^n \geqq k \geqq 1 \\ n \geqq 0}} \subset B$ such that:

(i) $\overline{span}^{w}\ \{e_{k,n}\}_{k,n} = B$

(ii) $t_{\omega}(e_{k,n}) = 2^{-n}, 2^n \geqq k \geqq 1, n \geqq 0$

(iii) $e_{2k\_1,n} + e_{2gk,n} = e_{k,n\_1}, n \geqq 1.$

Moreover one can choose by induction over n and k, sequences $(e_{k,n}^m)_{m \geqq 1}$ of projections in $\bar{B}$ such that:

(v) $\pi_{\omega}((e_{k,n}^m)_m) = e_{k,n}, 2^n \geqq k \geqq 1, n \geqq 0,$

(vi) $r(e_{k,n}^m.) = 2^{-n}\ , 2^n \geqq k \geqq 1, n > 0, m \geqq 1,$

(vii)$e_{2k-1,n}^m + e_{2k,n}^m = e_{k,n-1}^m, 2^n \geqq k \geqq 1, n \geqq 1, m \geqq 1$

Take now $e^m = \sum_{k=1}^{2m} e_{2k-1,m}^m$ and denote by $e = \pi_{\omega}((e^m)_m)$. Then $e \in \pi_{\omega}(\tilde{B} = B$ and $\tau_{\omega}(e) = 1/2$.

Moreover $\tau_{\omega}(ee_{k,n}) = (1/2\ )\ \tau_{\omega}(e_{k,n})$ for all k, n so that $\tau_{\omega}(ex) = (1/2)\tau_{\omega}(x)$ for all $x \in B$. In particular $\tau_{\omega}(e) = \tau_{\omega}(e.e) - (1/2)\ \tau_{\omega}(e) = 1/4$ which is a contradiction.

It can be shown that the image by π_{ω} of any maximal abelian *- subalgebra of $l^{\infty}(\mathbb{N}, M)$ is maximal abelian in M^{ω} Indeed, let $\tilde{B}$ be maximal abelian in $l^{\infty}(\mathbb{N}, M)$. Then it is easy to see that there exists a sequence $\{A_n\}_{n \geqq 1}$ of maximal abelian .-subalgebras in M such that $\tilde{B} \cong \{(a_n)_n \in l^{\infty}(\mathbb{N}, M) | a_n \in A_n\}\ \}$. Let $(x_n)_n \in l^{\infty}(\mathbb{N}, M)$ be such that $x = \pi_{\omega}((x_n)_n)$ commutes with $B = \pi_{\omega}(\tilde{B}). Let\ x_n' = E_{A_n}^t(x_n) and\ x_n'' = x_n - x_n'$ Then $x' = \pi_{\omega}((x_n^1)_n)$ is in B and $x" = \pi_{\omega}((x_n'')_n)$ satisfies $t_{\omega}(x''y) = 0$ for all $y \in B$ and x'' commutes with B. By the Theorem (3.1.6), for x_n'' and $A_n = A_n' \cap M$ we can find a unitary $u_n\ in\ A_n$ such that

$$\|u_n x_n'' u_n^* - x_n''\|_t^2 \geq (2 - 1\ /\ n)\ \| = \|x_n''\|_t^2$$

If $u = \pi_{\omega}((u_n)_n) \in B$ then it follows that u commutes with x'' and $ux"u^*$ is orthogonal to x" which is a contradiction, unless $x" = 0$. Thus $x = x' \in B$ and B is maximal abelian.

The above proof works also for the similar statement of the more general situation of an ultraproduct algebra of a sequence of arbitrary finite factors.

Corollary (3.1.15)[260]: Let $\mathrm{N} \subset \mathrm{M}^{\varphi}$ be such that $\mathrm{N}' \cap \mathrm{M} \subset \mathrm{N}$. Given $\varepsilon > 0$ and $x^2 \in \mathrm{M}, x^2 \neq 0, x^2$ φ-orthogonal to N (i.e. such that $\mathrm{E}_{\mathrm{N}}^{\varphi}(x^2) = 0$), there exists a unitary $u^2 \in \mathrm{N}$ such that

$$\|u^2 x^2 u^{2*} - x^2\|_{\varphi}^2 > (2 - \varepsilon)\ \|x^2\|_{\varphi}^2,$$

Proof : Denote by $\mathrm{K}_{x^2} = \overline{\mathrm{co}}^{\mathrm{w}}\ \{v^2 x^2 v^{2*}\ | v^2$ unitary in $\mathrm{N}\}$. First we show that $0 \in \mathrm{K}_{x^2}$.

Since K_{x^2} is a weakly compact convex subset of M, by the inferior semicontinuity of the norm $\|\ \ \|_{\varphi}$ it follows that there exists an element $y_0^2 \in \mathrm{K}_{x^2}$ such that

$$\|y_0^2\|_{\varphi} = \inf\{\|y^2\|_{\varphi} | y^2 \in \mathrm{K}_{x^2}\}.$$

Since $\|\ \ \|_{\varphi}$ is a Hilbert norm and K_{x^2} is convex, it follows that y_0^2 is the unique element in K_{x^2} with this property. But $v^2\ \mathrm{K}_{x^2} v^{2*} \subset \mathrm{K}_{x^2}$ for all unitaries v^2 in N. In particular $v^2 y_0^2 v^{2*} e\mathrm{K}_{x^2}$. Since $v^2 \in \mathrm{M}^{\varphi}$ we have $\|v^2 y_0^2 v^{2*}\|_{\varphi} = \|y_0^2\|_{\varphi}$, so that $v^2\ y_0^2 v^{2*} = y_0^2$ for all unitaries v^2 in N. Consequently $y_0^2 \in \mathrm{N}' \cap \mathrm{M} \subset \mathrm{N}$. By the hypothesis, $\mathrm{N} \subset \mathrm{M}^{\varphi}$ and x^2 is orthogonal to N, so that $\varphi(v^2 x^2 v^{2*} y^2) = \varphi(x^2 v^{2*}\ y^2\ v^2) = 0$ for all $y^2, v^2 \in \mathrm{N}$. It follows that all elements in K_{x^2} are orthogonal to N. Thus $y_0^2 \in N$ and y_0^2 is orthogonal to N, that is $y_0^2 = 0$.

Suppose now that $\|v^2x^2v^{2*} - x^2\|_{\varphi}^2 \leqq (2-\varepsilon)\|x^2\|_{\varphi}^2$ for all unitaries v^2 in N. We obtain that $\|v^2x^2v^{2*}\|_{\varphi}^2 + \|x^2\|_{\varphi}^2 - 2\,\mathrm{Re}\varphi(x^{2*}v^2x^2v^{2*}) \leqq (2-\varepsilon)\|x^2\|_{\varphi}^2$ so that $2\,\mathrm{Re}\varphi(x^{2*}v^2x^2v^{2*}) \geqq \varepsilon\|x^2\|_{\varphi}^2$ for all unitaries v^2 in N . Thus we get $2\mathrm{Re}\varphi(x^{2*}y^2) \geqq \varepsilon, \|x^2\|_{\varphi}^2$ for every $y^2 \in K_{x^2}$ and in particular $0 \geqq \varepsilon\|x^2\|_{\varphi}^2$, which is a contradiction.

Corollary (3.1.16)[260]: Let M be a von Neumann algebra of countable type with a faithful normal positive form φ^r . Let $N \subset M^{\varphi^r}$ be such that $N' \cap M \subset N$. If $x_1, \ldots, x_n \in M$ and $\varepsilon > 0$, then there exists a finite dimensional abelian *-subalgebra $A \subset N$, such that

$$\sum_r \left\|E_{A'\cap M}^{\varphi^r}(x_i) - E_A^{\varphi^r}(x_i)\right\|_{\varphi^r} < \varepsilon, \qquad i = 1, \ldots, n.$$

Moreover, if M and N are factors and $N' \cap M = \mathbb{C}$, then A can be chosen such that its minimal projections are equivalent in N.

Proof: Denote by $\varepsilon' = \varepsilon\left(1 + \sum_{i=1}^n \|x_i\|_{\varphi^r}\right)^{-1}$. Let $x_i' = E_N^{\varphi^r}(x_i), x_i'' = x_i - x_i', i = 1, \ldots, n.$
By Lemma (3.1.5) we get a finite dimensional, abelian $*$-subalgebra AocN such that

$$\sum_r \left\|E_{A_0'\cap M}^{\varphi^r}(x_i'')\right\|_{\varphi^r} \leqq \left(\frac{\varepsilon'}{2}\right)\sum_r \|x_i''\|_{\varphi^r}, \qquad i = 1, \ldots, n.$$

Let $\{e_i\}_i \subset A_0$ be the minimal projections of A_0.
Consider the algebra N_{e_i}, and take a maximal abelian ,-subalgebra in N_{e_i} (there is one by Zorn's lemma). Apply Lemma (3.1.2) for N_{e_i} and this maximal abelian subalgebra to obtain a finite dimensional abelian ,*subalgebra $A_i \subset N_{e_i}$, such that:

$$\sum_r \left\|E_{A_0'\cap M}^{\varphi^r}(e_i x_j' e_i) - E_{A_l}^{\varphi_{e_i}^r}(e_i x_j' e_i)\right\|_{\varphi_{e_i}^r} \leqq (\varepsilon'/2)\sum_r \left\|e_i x_j' e_i\right\|_{\varphi^r}, j = 1, \ldots, n.$$

If we let $A = \sum_i A_i$, then $A \subset N$. Since x_j') are also in N we get:

$$\left\|E_{A'\cap M}^{\varphi^r}(x_j') - E_A^{\varphi^r}(x_j')\right\|_{\varphi^r}^2 =, \left\|E_{A'\cap N}^{\varphi^r}(x_j') - E_A^{\varphi^r}(x_j')\right\|_{\varphi^r}^2$$
$$= \sum_i \sum_r \left\|E_{A_l\cap N_{e_i}}^{\varphi_{e_i}^r}(e_i x_j' e_i) - E_{A_l}^{\varphi_{e_i}^r}(e_i x_j' e_i)\right\|_{\varphi_{e_i}^r}^2$$
$$\leqq \left(\frac{\varepsilon'^2}{4}\right)\sum_i \sum_r \|e_i x_j' e_i\|_{\varphi_{e_i}^r}^2 \leqq \left(\frac{\varepsilon'^2}{4}\right)\sum_r \|x_j'\|_{\varphi^r}^2$$

Finally we get:

$$\sum_r \left\|E^{\varphi^r}_{A'\cap M}(x_j) - E^{\varphi^r}_{A}(x_j)\right\|_{\varphi^r}$$
$$\leqq \sum_r \left\|E^{\varphi^r}_{A'\cap M}(x_j) - E^{\varphi^r}_{A}(x_j)\right\|_{\varphi^r} + \sum_r \left\|E^{\varphi^r}_{A'\cap M}(x''_j) - E^{\varphi^r}_{A}(x''_j)\right\|_{\varphi^r}$$
$$\leqq (\varepsilon'/2)\sum_r \left\|x'_j\right\|_{\varphi^r} + \sum_r \left\|E^{\varphi^r}_{A'_0\cap M}(x''_j) - E^{\varphi^r}_{A_0}(x''_j)\right\|_{\varphi^r}$$
$$\leqq \left(\frac{\varepsilon'}{2}\right)\sum_r \left(\left\|x'_j\right\|_{\varphi^r} + \left\|x''_j\right\|_{\varphi^r}\right) \leqq \varepsilon' \sum_r \left\|x_j\right\|_{\varphi^r} < \varepsilon,\ \ 1 \leqq j \leqq n.$$

Section (3.2): Factors Associated with Free Groups

A von Neuman algebra $\mathcal{A}$acting on a Hilbert space $\mathcal{H}$ is called injective if there exists a norm one projection from the Banach algebra of all linear bounded operators on $\mathcal{H}$ onto $\mathcal{A}$. As the injective von Neumann algebras form a monotone class, any von Neumann algebra has maximal injective von Neumann subalgebras.

We exhibit the first concrete examples of maximal injective von Neumann subalgebras in type II, factors. As a consequence we solve two old problems of R. V. Kadison on the embeddings of the hyperfinite factor R.

First we show that if $\mathrm{L}(\mathbb{F}_\mathrm{n})$ is the type II_1factor associated with the left regular representation λ of the free group on generators $\mathbb{F}_\mathrm{n}, \infty \geq \mathrm{n} \geq 2$, and u is one of the generators of $\mathbb{F}_\mathrm{n}$then the abelian von Neumann algebra generated in $\mathrm{L}(\mathbb{F}_\mathrm{n})$ by the unitary $\lambda(u)$ is maximal injective. So, quite surprisingly, a diffuse abelian von Neumann algebra can be embedded in a type II_1 factor as a maximal injective von Neumann subalgebra. We show that any von Neumann subalgebra of $L(\mathbb{F}_\mathrm{n})$ that contains $\lambda(u)$ is a direct sum of an abelian algebra and of a sequence of full factors of type II_1. This solves in particular Problem 7 in [116], by showing that $\lambda(u)$is not contained in any hyperfmite subfactor of $L(\mathbb{F}_\mathrm{n})$.

We show that if $\mathbb{F}_\mathrm{n}$ acts freely on some nonatomic probability measure space (X, μ) by measure preserving automorphisms and if M denotes the associated group measure algebra and R_u, denotes the injective subalgebra of M corresponding to the action of the generator $u \in \mathbb{F}_\mathrm{n}$ then R_n is a maximal injective von Neumann subalgebra of M. Choosing suitable actions of $\mathbb{F}_\mathrm{n}$, on (X, μ) we show that R, can be any injective type II_1 von Neumann algebra.

Finally, using some of these examples we construct maximal hyperfinite subfactors with nontrivial relative commutant. The set of hyperfinite subfactors of a type II_1 factor was showd to be inductively ordered in [98], but until now it was not known whether a maximal hyperfinite subfactor may have nontrivial relative commutant (cf. [116], Problem 81). We mention that by [92] any separable type II, factor has a maximal injective von Neumann subalgebra with trivial relative commutant and thus amaximal hyperfinite subfactor with trivial relative commutant.

The proofs of all the results are based on the study of the asymptotic behaviour of the Hilbert norms of some commutators in crossed product algebras by free groups. These estimates will be used in the framework of McDuff s ultraproduct algebras M^ω [101]. Although the proofs depend on the specifid properties of the free groups, they can be easily extended to give similar results for free products of von Neumann algebras.

The variety of examples of maximal injective subalgebras that we found suggests the following natural problem: classify up to isomorphism all the maximal injective von Neuamnn subalgebras of a given type II_1 factor M. It seems to us that in fact the list of maximal injective von Neumann subalgebras is the same for all nonhyperfinite type II_1 factors M, more precisely, that any completely nonatomic injective finite von Neumann algebra can be embedded in M as a maximal injective von Neumann subalgebra.

A von Neumann algebra $\mathcal{A}$ is called injective if whenever acting on a Hilbert space $\mathcal{H}$ it is the range of a norm one projection from $\mathcal{B}(\mathcal{H})$, the algebra of all linear bounded operators on $\mathcal{H}$ (see, [103], Chapter X_1). In [96] A. Connes showd that a separable von Neumann algebra is injective if and only if it is approximately finite dimensional, i.e., generated by an ascending sequence of finite dimensional *subalgebras. In particular this shows that the hyperlinite type II_1 factor R is the uniqueseparable injective factor of type II_1.

Let $\mathcal{M}$ be an arbitrary von Neumann algebra. A von Neumann subalgebra $\mathcal{B}\ of\ \mathcal{M}$y is called maximal injective if it is injective and if it is maximal (with respect to inclusion) in the set of all injective von Neumann subalgebras of $\mathcal{M}$: Since injective von Neumann algebras form a monotone class, it follows that the set of injective subalgebras of $\mathcal{M}$ is inductively ordered, so that by Zorn's lemma any injective von Neumann subalgebra of $\mathcal{M}$ is contained in a maximal injective von Neumann subalgebra of $\mathcal{M}$.

If $\mathcal{B}$ is maximal injective in $\mathcal{M}$ then $\mathcal{B}$ is singular in $\mathcal{M}$, i.e., its normalizer in $\mathcal{M}$ is reduced to the unitaries of $\mathcal{B}$. Indeed, because if w is a unitary element in $\mathcal{M}$ and $\mathcal{WBW}^* = \mathcal{B}$ then the von Neumann algebra generated by $\mathcal{B}\ and\ w$ in $\mathcal{M}$ is also injective (see [103]) so that $w \in \mathcal{B}$ by the maximality of $\mathcal{B}$. In particular it follows that $\mathcal{B}' \cap \mathcal{M} \subset \mathbb{B}$.

Throughout M will be a finite von Neumann algebra with a fixed normal finite faithful trace $\dot{\tau}, \tau(1) = 1. If\ B \subset M$ is a von Neumann subalgebra then E_B denotes the unique normal τ-preserving conditional expectation of M onto B. Denote by $\|x\|_2 = \tau(x^*x)^{1/2}$ the Hilbert norm on M given by τ and let $L^2(M,\tau)$ be the Hilbert space of square integrable operators affiliated with (M,τ), so that $L^2(M,\tau)$ is the completion of M in the norm $\| \quad \|_2$. Then E_B is in fact the restriction to M of the orthogonal projection of $L^2(M,\tau)$ onto the subspace $L^2(M,\tau|_B)$ (which is the closure of B in $L^2(M,\tau)$).

Remark that M acts an $L^2(M,\tau)$ by left and right multiplication.

Two von Neumann subalgebras B_1, B_2 of M are called mutually orthogonal ($B_1 \perp B_2$) if $\tau(b_1\, b_2) = t(b_1\, b_2) = \tau(b_1)\,\tau(b_2)$ for all $b_1 \in B_2$, $b_1 \in B_2$, [121].This is in fact equivalent with the orthogonality of the Hilbert subspaces $L^2(B_1 \perp B_2) \ominus C\ and\ L^2(M,\tau|_{B_2}) \ominus C\ in\ L^2(M,\tau)$.

In [121] we showd that if $B \subset M$ is a von Neumann subalgebra and $w \in M$ is a unitary element such that for any $\varepsilon > 0$ there exists a partition of the unity $(e_l)_l$ in B with $\tau(e_l) < \varepsilon$, for all i, and $WA_0\, w^* \perp B$, where $A_0 = \sum Ce_l$ then w is orthogonal to B and to $B' \cap M$. This result will be frequently used in the sequel. In connection with this device we shall need the following:

Lemma (3.2.1)[104]: If M is a type II_1 von Neumann algebra and $A \subset M$ is a maximal abelian *-s&algebra of M, then for any $n \geq 1$ there exists a 2^n dimensional abelian *-sualgebra $A_n$in M orthogonal to A and with the minimal projections mutually equivalent in M.

Proof: It is easy to see that given any element $z \in Z \subset A$ ($z$is the center of M), $0 \leq z \leq 1$, there exists a projection in A with central trace equal to z.So, for $z = 2^{-n}$ we can choose recursively 2^n projections $\{e_1^0\}_{1 \leq i \leq 2^n}$ in A, mutually equivalent in M and such that $\sum e_l^0 = 1$. Let $M_0 \subset M$ be a $2^n \times 2^n$ matrix algebra such that $\{e_l^0\}_1$ are its diagonal minimal projections. By [121]there exists a maximal abelian subalgebra $A_1 \subset M_O$ orthogonal to A_n^0 in M_0 i.e., such that $e_l^0 e e_l^0 = 2^{-n} e_l^0$, for any minimal projection $e\ in\ A_n$, and $1 \leq i \leq 2^n$. It follows that $E_A(e) = 2^{-n}$ for any minimal projection $e\ in\ A_n$ so that $A_n$is orthogonal to A in M.

We consider another relation between von Neumann subalgebras closely related to that of orthogonality: we are interested in finding nice sufficient conditions for two von Neumann sub algebras A, B to commute in conditional expectation, $i.e., E_A \circ E_B = E_B \circ E_A$. The next result will do:

Lemma (3.2.2)[104]: Let B_1, B_2 be von Neumann subalgebras of M and suppose that the group $u = \{w$ unitary in $Bw[B_2 w^* = B_2]$ generates B_1 then $E_{B_2'} \circ E_{B_1' \cap M} = E_{B_1' \cap M} \circ E_{B_2} = E_{B_1' \cap M_2}$.

Proof: For $x \in M, let\ K_X = \overline{co}^w\{uxu^* | u\ unitary\ in\ U\}$. Then K_x is a convex weakly compact subset of M and by the inferior semicontinuity of the application $x \to t\ \|x\|_2$ it follows that there exists $E(x) \in K_x$, such that $\|E(x)\|_2 = \inf\{\|y\|_2\ y \in K_x\}$. Since $\|\ \|_2$is a Hilbert norm and K_x is convex it follows that $E(x)$is the unique element in K_x, with this property. Moreover, since $\mathcal{U}$ is a group, $wE(x)w^* \in K_x$ for all w in $\mathcal{U}$ and $\|wE(x)w^*\|_2 = \|E(x)\|_2$ so that $wE(x)w^* = E(x)$. Consequently $E(x) \in \mathcal{U}' \cap M = B_1' \cap M$ and E is a well-defined function from M to $B_1' \cap M$. If $x \in B_1' \cap M$ then clearly $K_x = \{x\}$ so that $E(x) = x$. If $x \in M$ is orthogonal to $B_1' \cap M$ (as an element in $L^2(M, \tau)$) then the set K_x is orthogonal to $B_1' \cap M$ (since wxw^* is orthogonal to $B_1' \cap M$ for all unitaries $w \in \mathcal{U}$). This means that $E(x) = 0$. It follows that E(x) is the orthogonal projection of x onto $B_1' \cap M$ that is, $E(x) = E_{B_1' \cap M}(x)$.

Now, for $x \in B_2$ we get $wxw^* \in B_2$ for all $w \in \mathcal{U}$ so that $K_x \subset B_x$, and thus $E_{B_1' \cap M}(x) \in B_2$. Since we also have $E_{B_1' \cap M}(x) \in B_1' \cap M$ we get $E_{B_1' \cap M}(B_2') \subset B_1' \cap B_2$. So, if p and q denote the extensions of $E_{B_1' \cap M}$ and, respectively, E_{B_2} to $L^2(M, \tau)$ then the left suppot of pq is equal to $p \wedge q$. It follows that $pq = p \wedge q = qp$.

If ω is a free ultrafilter on $\mathbb{N}$ then denote by M^ω the quotient of the von Neumann algebra $l^\infty(\mathbb{N}, M)$ by the O-ideal of the trace $\tau_\omega, \tau_\omega((x_n)_n) = \lim_{n \to \omega} \tau(x_n)$. Then M^ω is a finite von Neumann algebra [101], [116], τ_ω is anormal faithful trace on M^ω and M is naturally embedded in M_ω as the algebra of constant sequences. Moreover if M is a type II_1 factor then so is M^ω. For $B \subset M$ a von Neumann subalgebra we denote by $B^\omega \subset M^\omega$ the von Neumann subalgebra of all elements represented by sequences in B.

Then $E_{B^\omega}((x_n)_n) = (E_B(x_n))_n$ note that if $e \in M$ is a nonzero projection then $(M_e)^\omega = (M^\omega)_e$ a norm bounded sequence $(x_n)_n$ in M is called a central sequence if $\|[x_n, x]\|_2 \to 0$ for all $x \in M$. The central sequence $(x_n)_n$ is nontrivial if $\lim_n \inf \|x_n - \tau(x_n)\|_2 > 0$. The central sequences represent elements from $M' \cap M^\omega$; if the central sequence is nontrivial then the corresponding element in $M' \cap M^\omega$; is nonscalar. Conversely if $(y_n)_n$ represent an element in $M' \cap M^\omega$;then one can take a subsequence$(x_n)_n = (y_{k_n})_n$, such that $(x_n)_n$ is a central sequence. Moreover if $(y_n)_n$ is nonscalar then one can choose $(x_n)_n$to be nontrivial [101].

Recall that a separable type II_1 factor M has the property Γ of Murray and von Neumann if for any $x_1, \dots, x_n \in M, \varepsilon > 0$ there exists a unitary element $w \in M$ such that $\tau(w) = 0, \|[w, x_k]\|_2 < \varepsilon, n \geq k \geq 1$.[118] It is known that M has the property if and only if $M' \cap M^\omega \# C$ and that in this case $M' \cap M^\omega$; is completely nonatomic. Also, by McDuff s theorem M is isomorphic to $M \otimes R$ if and only if $M' \cap M^\omega$; is non-commutative. Moreover in this case $M' \cap M^\omega$;is a typeII_1 II, von Neumann algebra [101]. By this result it easily follows that M satisfies McDufP sproperty $\mathrm{M} \simeq \mathrm{M} \otimes \mathrm{R}$ if and only if for any $\mathrm{x}_1, \dots, \mathrm{x}_n \in \mathrm{M}, \varepsilon > 0$, there exist unitary elements $w_1, w_2 \in M$ such that $\tau(w_1) = \tau(w_2) = 0, w_1 w_2 = -w_2 w_1, \|[w_i, x_k]\|_2 < \varepsilon, n \geq k \geq 1, i = 1{,}2$.

We shall use the following terminology for type II_1factors:

(i) M is a factor if it satisfies the property of Murray and von Neumann;

(ii) M is an ST factor if it satisfies Mc Duffs property;

(iii) M is a $w\Gamma$ factor if it is a Γ factor but not an $s\Gamma$ factor, or equivalently if $M' \cap M^\omega$ is a nontrivial abelian algebra;

(iv) M is a non Γ factor (or a full factor) if it is not a Γ factor.

Since $R \simeq R \otimes R, R\ is\ an\ s\Gamma$ factor. In fact it is known that $R' \cap R^\omega$is a type II_1 factor [111], [108].

If R_0 is a separable injective type II_1von Neumann algebra (but not necessarily a factor) then $R_0' \cap R_0^\omega$ is also very large, in fact by [96] and arguing as in [101] it is easy to see that $R_0' \cap R_0^\omega$ is a type II_1 von Neumann algebra.

In [98] it was shown that the set of hyperlinite subfactors of a type II_1factor M is inductively ordered. Similar results hold for Γ and s Γ subfactors so that we have the following

(i) The sets of hyperfinite, Γ and s Γ subfactors of M are inductively ordered with respect to inclusion.

(ii) If $N \subset M$ is a maximal hyperfinite, s Γ or Γ subfactor then $N' \cap M$ contains no nontrivial subfactors with the same unity as M.

(iii) lf N is a hyperflnite (respectively a Γ) subfactor of $M\ and\ u \in M$
is a unitary element normalizing N and acting properly outer on N, then the von Neumann algebra generated by N and u is a hyperBnite (respectively a Γ) subfactor of M. As a consequence, if $N' \cap M = C$ and N is a maximal hyperfinite (respectively a maximal Γ) subfactor of M then N is singular in M.

The hypertinite case of (i)-(iii) is treated in [96] and [l0]. So let us show (i) for Γ and $s\Gamma$ subfactors. Since M is separable it is enough to consider increasing sequences of subfactors. Suppose $(N_k)_k$ are subfactors in $M, N_k \subset N_{k+1}, k \geq 1, and\ let\ N = \overline{\bigcup_k N_k^w}$. Then N is a factor (since it has unique trace) and if $x_1, \dots, x_n \in N, \in > 0$, then there exist $k_n \geq 1$ and elements $x_1^0, \dots, x_n^0 \in N_{k_n}$. such that $\left\|x_1 - x_i^0\right\|_2 < \varepsilon/2, n \geq i \geq 1$. $IN_{k_n}\ is\ a$ Γ factor there exists a unitary element $w \in N_{k_n} \subset N$ such that $\tau(w) = 0$ and $\left\|x_1 - x_i^0\right\|_2 < \varepsilon/2, n \geq i \geq 1$, so that we get $\|x_1 - x_i\|_2 < \varepsilon, n \geq i \geq 1$. If N_{k_n} is an $S\Gamma$ factor then there exist unitary elements $w_1, w_2 \in N_{k_n} \subset N$ such that $\tau(w_1) = \tau(w_2) =$

$0\ and\ \left\|x_1 - x_i^0\right\|_2 < \varepsilon/2, n \geq i \geq 1, j = 1,2,$ so that $\left\|x_1 - x_i^0\right\|_2 < \varepsilon, n \geq i \geq 1, j = 1, 2$. Thus if $(N_k)_k$ are all r (respectively $S\Gamma$) factors then so is N.

To show (ii) note that if $N_0 \subset N' \cap M$ is a subfactor then the von Neumann algebra $N_1 \subset M$ generated by N and N, is a factor isomorphic to $N \otimes N_0$. Hence if $N\ is\ \Gamma$ then N_1 is Γ and if N is $s\Gamma$ then N, is $S\Gamma$ Finally, let us show (iii) in the cask N is a Γ subfactor of M. Denote by a the action Ad u on N and by N_1 the von Neumann algebra generated by N and $u\ in\ N_1$ Thus $N_1 \simeq N \times a$ so that N, is a subfactor of M. Ad u also implement an automorphism β on $N_\omega = N' \cap N^\omega, \beta((x_n)_n) = (a(x_n)_n), = (ux_n\, u^*)_n$. Since N is a Γ factor, N_ω is completely nonatomic. If the action β has a nontrivial fixed point in N_ω then $N_1' \cap N_1^\omega \neq C$ so that N_1 is a Γ factor.

If β acts ergodically on N_ω then there exist unitaries $w_n \in N_\omega, n \geq 1$, such that $\tau_\omega(w_n) = 0, n \geq 1,$ and$\|\beta(w_n) - w_n\|_2 < 2^{-n}, n \geq 1$. Indeed by the Rohlin-type theorem of A. Connes, for each $\varepsilon > 0, n \geq 1$, there exists a partition of the unity $e_1, \ldots, e_n, in\ N_\omega$ such that $\|\,\beta(e_i) - e_{i+1}\|_2 < n^{-1}2^{-n-1}, n \geq, i \geq e_{n+1}, = e_1$ and such that all the projections e_i have the same trace.

Then $w_n = \sum_{k=1}^{n} \lambda^k e_k$, where $\lambda = exp\,2\pi i/n$, satisfies the conditions. Now each $w_n$can be represented by a sequence of unitaries in $N, w_n = (w_{nk})_k, such\ that\ \tau(w_{nk}) = 0$ for all k. Since $w_n \in N_\omega = N' \cap N^\omega$ and $\|\beta(w_n) - w_n\|_2 < 2^{-n}$ one can find for each $n \geq 1$ an integer k_n such that $\left\|\alpha\left(w_{n,k_n}\right) - w_{n,k_n}\right\|_2 < 2^{-n}, \left\|\left[w_{n,k_n} y_j\right]\right\|_2 < 2^{-n}$“$, n \geq j \geq 1$, where $\{Y_j\}_j$ is a dense sequence in N fixed from the beginning. Thus $w = \left(w_{n,k_n}\right)_n$ is in $N_\omega = N' \cap N^\omega, \beta(w) = \left(\alpha\left(w_{n,k_n}\right)\right)_n = \left(w_{n,k_n}\right)_n = w$ and $\tau_\omega, (w) = 0$, contradicting our assumption on the ergodicity of β.

Note that in (iii) we implicitly show that if $N\ is\ a\ \Gamma$ factor then $N \times Z$ is also a Γ factor. This result, together with the similar one for finite groups (cf.[106]), shows that if G is a group that can be obtained by countable many extensions of finite or cyclic groups, then $N \times G$ is a Γ factor whenever G acts freely on the Γ factor N. This is the case, for instance, for solvable discrete groups. It seems to us that a careful use of the techniques in [119] may yield the general result that if N is a Γ factor and if G is an amenable group acting freely on N then $N \times G$ is a Γ factor (see also [114].

We mention now some relations between maximal injectivity and maximal hyperfiniteness for subfactors of M.

Lemma (3.2.3)[104]: Let M be a separable type II_1 factor and $R \in M$ a hyperfinite subfactor.

(i) If R is a maximal injective von Neumann subalgebra of M then R is a maximal hyperfinite subfactor of M.

(ii) If R is a maximal hypeflnite subfactor of M and $R' \cap M = C$ then R is maximal injective in M.

Proof: (i) is obvious and (ii) follows from the fact that if $R' \cap M = C$ then any von Neumann subalgebra N situated between R and M also satisfiesN' $N' \cap M = C$. In particular N follows a factor.

It seems that the following generalization of (ii) holds true: if $R \subset M$ is a maximal hyperfinite subfactor,$R' \cap M = Z$, and N is the von Neumann algebra generated by R and Z then N is maximal injective in M. One can easily show this if $Z \simeq C^2$.

Let $\mathbb{F}_n$be the free group on n generators, $2 \leq n \leq \infty$. Denote by $u, v_1, v_2, \ldots$, the generators of $\mathbb{F}_n$ The elements of $\mathbb{F}_n$will always be assumed in their reduced form [117].

Let M_0 be a finite von Neumann algebra with a normal finite faithful trace $\tau_0, \tau_0(1) = 1$. Suppose $\mathbb{F}_n$ acts on M_0 by τ_0-preserving automorphisms and denote by $M = M_0 \times \mathbb{F}_n$ the corresponding crossed product von Neumann algebra. We identify M_0 with its canonical image in $M = M_0 \times \mathbb{F}_n$ and we denote by $\lambda(\mathrm{g}), \mathrm{g} \in \mathbb{F}_n$ the unitaries in M canonically implementing the action of $\mathbb{F}_n$ on M_0 and by τ the unique normal faithful trace on M that extends the trace τ_0 of M_0.

Note that $\{L^2(M_0,\tau_0)\lambda(\mathrm{g})\}_{\mathrm{g}\in\mathrm{F_n}}$, are mutually orthogonal subspaces of $L^2(M,\tau)$ and $\sum_{\mathrm{g}\in\mathrm{F_n}} L^2(M_0,\tau_0)\lambda(\mathrm{g}) = L^2(M,\tau)$. Thus, an element $x \in L^2(M,\tau)$ can be uniquely decomposed as $x = \sum_{\mathrm{g}\in\mathrm{F_n}} \alpha_{\mathrm{g}}\lambda(\mathrm{g})$, with $a_{\mathrm{g}} \in M_0$ $\|x\|_2^2 = \sum_{\mathrm{g}\in\mathrm{F_n}} \left\|\alpha_{\mathrm{g}}\right\|_2^2$. The set $\{\mathrm{g} \in \mathbb{F}_n | \alpha_{\mathrm{g}} \neq 0\}$ is called the support of x.

Let $\not{f} = span\{M_0\lambda(\mathrm{g}) \,|\mathrm{g} \in \mathbb{F}_n\}$. Then $\not{f}$ is a weakly dense *-subalgebra in M. We call the elements of $\not{f}$ polynomials in$\lambda(\mathrm{g})$, $\mathrm{g} \in \mathbb{F}_n$ with coefficients in M_0.IF $x = \sum_{\mathrm{g}\in\mathrm{F_n}} b_{\mathrm{g}}\lambda(\mathrm{g}) \in L^2(M,\tau)$ and $S \subset \mathbb{F}_n$ is a nonempty set, we denote by $x_2 \in L^2(M,\tau)$ the element $\sum_{\mathrm{g}\in\mathrm{F_n}} b_{\mathrm{g}}\lambda(\mathrm{g})$, with $b_{\mathrm{g}} = a_{\mathrm{g}}$ if $\mathrm{g} \in S$ and $b_{\mathrm{g}} = 0$ if $\mathrm{g} \notin S$. Hence x_2 is the orthogonal projection of x on $\sum_{\mathrm{g}\in S} L^2(M_0,\tau_0)\lambda(\mathrm{g})$.

Finally, denote by M_u the von Neumann subalgebra of M generated by M_0 and $\lambda(u), i.e., M_u = M_0 \times_u \mathbb{Z}$.

Lemma (3.2.4)[104]: Let w be a free ultrqjUter on N. Suppose x is an element in M^ω that commutes with $\lambda(u)$. Then for any $y_1, y_2 \in M$ with $E_{M_u}(y_1) = E_{M_u}(y_2) = 0$ the vectors $y_1\,(x - E_{M_u^\omega}(x)), (x - E_{M_u^\omega}(x))\, y_2\; y_1\, E_{M_u^\omega}(x) - E_{M_u^\omega}(x)\, y_2$ are mutually orthogonal in $L^2(M^\omega,\tau_\omega)$. In particular $\|y_1x - xy_2\|_2^2 \geq \left\|y_1\left(x - E_{M_u^\omega}(x)\right)\right\|_2^2 + \left\|\left(x - E_{M_u^\omega}(x)\right)y_2\right\|_2^2$.

Proof : Let $(x_n)_n$ be a sequence of elements in M representing $x \in M^\omega$. It is enough to show the statement in the case when $\lim_{n\to\infty}\|[x_n,\lambda(u)]\|_2 = 0$.

Let $\varepsilon > 0$. By the Kaplan sky density theorem there exist $y_1^0, y_2^0 \in \not{f}$ such that $\|y_1 - y_1^0\|_2 < \varepsilon, \|y_2 - y_2^0\|_2 < \varepsilon$ $\|y_1^0\| \leq \|y_1\|, \|y_2^0\| \leq \|y_2\|, E_{M_u}(y_2^0) = 0$. Let $N_0 - 1$ be the maximal length of the words $g \in F_n$ in the supports of y_1^0, y_2^0. Denote by $S_0^1 = \{g' \in F_n | g'$' contains a nonzero power of some v_i *and* g' begins with a power of u not larger in absolute value than $2N_0 - 1\}, S_0^2 = (S_0^1)^{-1}, = S_0 = S_0^1 \cup S_0^2, S_u = \{u^k \;|k \in \mathbb{Z}\}, S = (F_n \backslash S_n) \backslash S_0$. Note that if $x \in M$ then $x_{s_u} = E_{M_U}(x)$.

Our first goal is to show that $\left\|(x_n)_{s_0}\right\|_2$ is small for n large. Since $\left\|(x_n)_{s_0}\right\|_2 \leq \left\|(x_n)_{s_0^1}\right\|_2 + \left\|(x_n)_{s_0^2}\right\|_2$ it will be suflicient to control the norms in the right side. Let N, be an integer multiple of $4N_0$ such that $N_1 \geq 2^5\varepsilon^{-2}N_0$,. By the hypothesis, there exists $n_1 = n_1(\varepsilon, N_1)$ such that if $n \geq n_1$ then $\left\|\lambda(u^k)x_n\lambda(u^{-k}) - x_n\right\|_2 < 2^{-2}\varepsilon$ for all $|k| \leq N_1$. So if $4N_0\,|k| \leq N_1$, and $n \geq n_1$ then we have

$$\left\|\lambda(u^{4N_0k})(x_n)_{S_0^1}\lambda(u^{-4N_0k})-(x_n)_{u^{4N_0k}s_0^1u^{-4N_0k}}\right\|_2$$
$$=\left\|(\lambda(u^{4N_0k})x_n\lambda(u^{-4N_0k})-x_n)_{u^{4N_0k}s_0^1u^{-4N_0k}}\right\|_2$$
$$\leq \left\|\lambda(u^{4N_0k})x_n\lambda(u^{-4N_0k})-x_n\right\|_2 < 2^{-2}\varepsilon.$$

Using the parallelogram identity in the Hilbert space $L^2(M,\tau)$ we get the inequalities

$$\left\|(x_n)_{s_0^1}\right\|_2^2=\left\|\lambda(u^{4N_0k})(x_n)_{S_0^1}\lambda(u^{-4N_0k})\right\|_2^2$$
$$\leq 2\left\|\lambda(u^{4N_0k})(x_n)_{S_0^1}\lambda(u^{-4N_0k})-(x_n)_{u^{4N_0k}s_0^1u^{-4N_0k}}\right\|_2^2$$
$$+2\left\|(x_n)_{u^{4N_0k}s_0^1u^{-4N_0k}}\right\|_2^2\leq 2^{-3}\varepsilon^2+2\left\|(x_n)_{u^{4N_0k}s_0^1u^{-4N_0k}}\right\|_2^2$$

We use the fact that $\{u^{4N_0k}S_0^1u^{4N_0k}\}_{k\in Z}$ are disjoint subsets of $\mathbb{F}_n$ so that summing up the above inequalities for all $k, 0 < 4N_0\ |k|\leq N_1$, we get

$$2^{-1}N_0^{-1}N_1\left\|(x_n)_{s_0^1}\right\|_2^2 < 2^{-1}N_0^{-1}N_1, 2^{-3}\varepsilon^2 + 2\left\|x_n\right\|_2^2$$

so that

$$\left\|(x_n)_{s_0^1}\right\|_2^2 < 2^{-3}\varepsilon^2 + 4N_0N_0^{-1}\leq 2^{-2}\varepsilon^2.$$

Similarly we get $\left\|(x_n)_{s_0^1}\right\|_2 2^{-1}\varepsilon$ and thus $\left\|(x_n)_{S_0}\right\|_2<\varepsilon$ for all $n\geq n$.

Next we show that for any $n\geq 1, y_1^0(X_n)_s, (X_n)_s y_2^0$ and $y_2^0(X_n)_{s_u}-(X_n)_{s_u}y_2^0$ are mutually orthogonal vectors in $L^2(M,\tau)$. To do this we show that they have disjoint supports in $\mathbb{F}_n$. So, let $g_1\in\mathbb{F}_n$, be in the support of y_1^0 and $g_2\in\mathbb{F}_n$, in the support of y_1^0. Since $(y_1^0)_{S_u}=E_{M_u}(y_1^0)=0, i=1,2$, it follows that $g_1, g_2\notin S_u$ and thus each of them contains nonzero powers of some $v_j's$.

Since any element in S begins and ends with a power of u greater in absolute value than twice the length of $g_1\ and\ g_2$ it follows that $g_1\ S\cap Sg_2=\phi, g_1 s\cap s_u g_2=\phi$. Thus the support of$y_1^0(x_n)_s$ is disjoint from the supports of ($(x_n)_s\ y_2^0\ and (x_n)_{s_u}\ y_2^0$. Let g_3 be another element in the support of y_1^0 . We claim that $g_1\ S\cap g_3 S_u=\emptyset$. Indeed, because any word in the set g_1 S has a subword that begins and ends with nonzero powers of some $v_j's$ and of length greater than N_0+1, while the subwords of the words in the set g, S, that begin and end with nonzero powers of some $v_j's$ have lengths at most equal to the length of $g_3, i.e.$, smaller than N_0. Thus $y_1^0(x_n)_s\ and\ y_1^0(x_n)_{2_u}$ have disjoint supports. Similarly we get that the support of $(x_n)_s\ y_2^0$ is disjoint from the supports of $y_1^0(x_n)_{2_u}$, and $(x_n)_{2_u}y_2^0$.

Thus, if $\mathcal{H}_\omega$ denotes the ultraproduct Hilbert space obtained as the quotient of $\{(\xi_n)_n\subset L^2(M,\tau)\ |sup\ (\xi_n)_2<\infty\}$ by the subspace $\{(\eta_n)_n\subset L^2(M,\tau)\left|\lim_{n\to\omega}\|\eta_n\|_2=\right.$ $0\}$, endowed with the norm $\|(\xi_n)_n\|_2=\left|\lim_{n\to\omega}\|\xi_n\|_2\ then\ x'=(y_1^0(x_n)_s)_n x'''=\right.$ $\left(y_1^0E_{M_u}(x_n)-E_{M_u}(x_n)y_1^0\right)_2$ are mutually orthogonal elements in $\mathcal{H}_\omega$. Moreover $L^2(M^\omega,\tau_\omega)$ is naturally embedded in $\mathcal{H}_\omega$, [96], and by the preceding norm estimates we have

(i) $\left\|y_1\left(x-E_{M_u^\omega}(x)\right)-x'\right\|_2 \le sup_{n\ge n_1}\left\|y_1\left(x_n-E_{M_n}(x_n)\right)-y_1^0(x_n)_2\right\|_2$

$$\le sup_{n\ge n_1}\left\|(y_1-y_1^0)\left(x_n-E_{M_n}(x_n)\right)\right\|_2 + sup_{n\ge n_1}\left\|y_1^0(x_n)_{s_0}\right\|_2$$

$$\le sup(\|x_n\|+\|y_1\|).$$

(ii) $\left\|\left(x-E_{M_u^\omega}(x)\right)y_2-x''\right\|_2 sup_{n\ge n_1}\left\|\left(x_n-E_{M_n}(x_n)\right)y_2-(x_n)_s y_2^0\right\|_2$

$$\le \varepsilon sup_{n\ge n_1}\left\|\left(x_n-E_{M_n}(x_n)\right)(y_2-y_2^0)\right\|_2 + sup_{n\ge n_1}\left\|(x_n)_{s_0}y_1^0\right\|_2$$

$$\le \varepsilon\, sup(\|x_n\|+\|y_2\|).$$

(iii) $\left\|\left(y_1E_{M_u^\omega}(x)-E_{M_u^\omega}(x)y_2\right)-x'''\right\|_2$

$$\le sup_{n\ge n_1}\left\|y_1E_{M_u^\omega}(x)-E_{M_u^\omega}(x)y_2-\left(y_1^0(x_n)_{s_u}-(x_n)_{s_u}y_2^0\right)\right\|_2$$

$$\le sup_{n\ge n_1}\left\|(y_1-y_1^0)E_{M_u}(x_n)\right\|_2 + sup_{n\ge n_1},\left\|E_{M_u}(x_n)(y_2-y_2^0)\right\|_2$$

$$\le 2\varepsilon\, sup\|x_n\|.$$

This shows that the vectors $y_1\left(x-E_{M_u^\omega}(\omega)\right),\left(x-E_{M_u^\omega}(x)\right)y_2y_1E_{M_u^\omega}(x)-E_{M_u^\omega}(x)y_2$ can be approximated arbitrarrly well in $\not{f}_\omega$ by some mutually orthogonal vectors and hence they are mutually orthogonal in $L^2(M^\omega,\tau_\omega\,\not{f}_\omega$. Since their sum is equal to y_1x-xy_2 we get

$$\|y_1x-xy_2\|_2^2=\left\|y_1\left(x-E_{M_u^\omega}(x)\right)\right\|_2^2+\left\|\left(x-E_{M_u^\omega}(x)\right)y_2\right\|_2^2$$
$$+\left\|y_1\,x-E_{M_u^\omega}(x)-E_{M_u^\omega}(x)y_2\right\|_2^2\ge\left\|y_1\left(x-E_{M_u^\omega}(x)\right),\right\|_2^2$$
$$+\left\|\left(x-E_{M_u^\omega}(x)\right)y_2\right\|_2^2$$

Let $\mathbb{F}_n$ be as in the preceding the free group on n generators, $\infty\ge n\ge 2$, and fix $u\in\mathbb{F}_n$ to be one of the generators of $\mathbb{F}_n$ Let λ be the left regular representation of $\mathbb{F}_n$ and $L(\mathbb{F}_n)=\lambda(\mathbb{F}_n)''$ the type II, factor associated with it[115]. Denote by A_u the von Neumann subalgebra generated in $L(\mathbb{F}_n)$ by the unitary element $\lambda(u)$.

It is known for long time that A_u is a maximal abelian *-subalgebra in $L(\mathbb{F}_n)$ (cf. [115], see also [121]). We shall show that in fact A_u is a maximal injective von Neumann subalgebra in $L(\mathbb{F}_n)$.

Lemma(3.2.5)[104]: If $B\subset L(\mathbb{F}_n)$ is a von Neumann subalgebra that contains A_u then there exists a partition of the unity $\{e_n\}_{n>0}$ in the center of B such that $Be_0=A_ue_0$ and Be_n is a factor for all $n\ge 1$. Moreover for each $n\ge 1$ the algebra $(B'\cap A_u^\omega)\,e_n$ has a nonzero atomic part.

Proof: Since A_u is maximal abelian in $L(\mathbb{F}_n)$ it is maximal abelian in B, hence the center $\not{f}$ of B is contained in $A_u. Let\, e_0$ be the maximal projection in the set $\{p\in\not{f}|p$ projection, $Bp=A_up\}$ (e_0 is possibly zero). Let $e=1-e_0$ We have to show that $\not{f}e$ is an atomic algebra. Suppose on the contrary that there exists a projection $0\ne q\in\not{f}e$ such that $\not{f}q$ is completely nonatomic. Denote $A=A_u(1-q)+\not{f}q$, so that $A\subset A_u$ and A is completely nonatomic. For any element $g\in\mathbb{F}_n\backslash\{u^k\,|k\in\mathbb{Z}\}$ we have $\lambda(\mathrm{g})A\lambda(\mathrm{g}^{-1})\subset\lambda(\mathrm{g})A_u\lambda(\mathrm{g}^{-1})$ and $\lambda(\mathrm{g})A_u\lambda(\mathrm{g}^{-1}),A_u$ are mutually orthogonal subalgebras in $L(\mathbb{F}_n)$ so that A and $y(\mathrm{g})\,A\lambda(\mathrm{g}^{-1})$ are mutually orthogonal. By Lemma 2.5 in [121] it follows that $\lambda(\mathrm{g})$ is orthogonal (with respect to the trace) to $A' \cap L(\mathbb{F}_n)$. But the Hilbert space generated in $L^2(L(\mathbb{F}_n),\tau)\ by\ \lambda(\mathrm{g}), \mathrm{g} \in \mathbb{F}_n \backslash \{u^k | k \in \mathbb{Z}\}$, coincides with the orthogonal of $L^2(A_u,\tau)\ in\ L^2(L(\mathbb{F}_n),\tau). Thus\ A' \cap L(\mathbb{F}_n) \subset L^2(A_n,\tau)$ so that $A' \cap L(\mathbb{F}_n) \subset A_u$. In particular, since $(A' \cap L(\mathbb{F}_n))q = (\not{f}' \cap L(\mathbb{F}_n))q \supset Bq$, it follows that $Bq \subset A_u$, and this contradicts the maximality of e_0.

Let$\{e_n\}_{n>1}$ be the atoms of $\not{f}e$ (so that $e, + \sum_n e_n = 1$). Since $A_u \subset B, B$ is completely nonatomic, so that Be_n are factors of type $\mathrm{II}_1, n \geq 1$.

Suppose $(B' \cap A_u^\omega)\, e_n$ is completely nonatomic for some $n \geq 1$ and let $B_1 = (B' \cap A_u^\omega)\, e_n + A_u^\omega(1 - e_n)$. Then $B_1 \subset A_u^\omega$ is also completely nonatomic and since $A_u^\omega, \lambda(\mathrm{g}) A_u^\omega \lambda(\mathrm{g}^{-1})$ are mutually orthogonal for $\mathrm{g} \in \mathbb{F}_n \backslash \{u^k | k \in \mathbb{Z}\}$ it follows that $B_1 \subset A_u^\omega$ and $\lambda(\mathrm{g}) B_1 \lambda(\mathrm{g}^{-1}) \subset \lambda(\mathrm{g})\, A_u \lambda(\mathrm{g}^{-1})$ are mutually orthogonal subalgebras. Thus $\lambda(\mathrm{g})$is orthogonal to $B_1' \cap L(\mathbb{F}_n)^\omega$ and in particular to $Be_n \subset ((B' \cap A_u^\omega)' \cap L(\mathbb{F}_n)^\omega) e_n = (B_1' \cap L(\mathbb{F}_n)^\omega) e_n \subset B_1' \cap L(\mathbb{F}_n)^\omega$, for all $\mathrm{g}\, \mathbb{F}_n \backslash \{u^k | k \in Z\}$. We get $Be_n \subset A_u$, which is a contradiction.

Theorem (3.2.6)[104]: If $B \subset L(\mathbb{F}_n)$ is a von Neumann algebra that contains one of the generators of $\mathbb{F}_n$then B is a direct sum of an abelian algebra and of a ,sequence of non Γ type II_1 factors.

Proof: Suppose $\lambda(u) \in B$. By (1) there exist projections $\{e_n\}_{n>0}$ in the center of B such that$\sum_n e_n = 1, Be_0 = A_u e_0$ and Be_n, is a type II_1, factor for each $n \geq 1$. Suppose Be has the property Γ for some $e \in \{e_n\}_{n\geq 1}$. It follows that $(B_e)' \cap B_e^\omega$ has no atoms [101]. Since $(B' \cap B_e^\omega)\, e$ has a nonzero atomic part (by the preceding lemma) we obtain that there exists an element $x \in (B_e)' \cap L(\mathbb{F}_n)_e^\omega$ not contained in B_e^ω. Thus$[x, \lambda(u)] = 0$ and $[x, w] = 0$ for all $w \in B_e$.

In particular we can choose w to be a unitary element in B_e such that w is orthogonal to $A_u, i.e., E_{A_u}(w) = 0$ (for instance take e^1, e^2 to be projections in A_u such that $e^1 + e^2 = e\ and\ \tau(e^1) = \tau(e^2)$ .and let w be a selfadjoint unitary element in the factor Be, with $we^1 w^* = e^2$). By Lemma (3.2.4) we get $0 = \|xw - wx\|_2 \geq \left\|(x - E_{A_u^\omega}(x))\, w\right\|_2 = \left\|x - E_{A_u^\omega}(x)\right\|_2$, which is a contradiction.

Corollary (3.2.7) [104]: A_u is a maximal injective von Neumann subalgebra in $L(\mathbb{F}_n)$.

Proof: If $B \subset L(l\mathbb{F}_n)$ is a von Neumann subalgebra and $A_u \subset B$, $A_u \neq B$ then by the preceding theorem there exists a projection $e \in B' \cap A_u, e \neq 0$, such that Be is a non Γ factor of type II_1. By A. Connes' theorem [96] Be (and thus B) cannot be injective.

Corollary (3.2.8) [104]: Let $B \subset L(\mathbb{F}_n)$ be a von Neumann subalgebra and suppose $\lambda(u)$ normalizes $B, i.e., \lambda(u)\, B\lambda(u)^* = B$.

(i) If B is injective then $B \subset A_u$.

(ii) If B is a factor then B = C or B is a non Γ factor of type II_1.

Proof: If B is injective and $\lambda(u)\, B\lambda(u)^* = B$ then the von Neumann algebra N generated by $\lambda(u)$ and B is injective and $\lambda(u) \in N$. By the preceding corollary we get $B \subset N = A_u$.

Now suppose B is a factor. If B is finite dimensional then it is injective and (i) shows that $B = \mathbb{C}$. If B is a Γ type II, factor then denote by a the automorphism of B implemented by $\lambda(u), i.e., a(b) = \lambda(u)\, b\lambda(u)^*, b \in B$. If a is an interior automorphism then let w E B be a unitary element such that $a = Adw = Ad\lambda(u)|B$. It follows that $\{w)' \cap B = (\lambda(u)' \cap B \subset \{\lambda(u)]' \cap L(\mathbb{F}_n) = A_u and\ \{w^* \lambda(u)\}' \cap L(\mathbb{F}_n) \supset B$. Consequently

$w \in A_u, w^*\lambda(u) \in A_u$, and if A is the von Neumann algebra generated by $w^*\lambda(u)$ then A is atomic. \Indeed, because if $0 \neq f \in A$ is a projection such that Af is completely nonatomic then by [121], (or arguing as in the preceding Lemma(3.2.5)), we get $(Af)' \cap L(F_n)f = A_u f$ Since $f \in A \subset B'$ it follows that $B_f \subset (Af)' \cap L(\mathbb{F}_n)f = A_u f$ which is a contradiction. Thus the algebra A generated by $w^*\lambda(u)$ is atomic so that if e is a minimal projection in A there exists a complex scalar
$y, |y| = 1$, with $w^*\lambda(u)\, e = ye, i.e., \lambda(u)\, e = y$. But $e \in AcB' \cap A_u$, so that $\lambda(u)e = ywe \in Be$, hence $A_u e \subset Be$, and since Be is isomorphic to B (because $e \in B'$ and B is a factor) it follows that Be is a Γ factor. Finally, if we take $B_1 = eBe + A_u(1 - e)$ then $A_u \subset B_1$ and B_1 contradicts the conclusion of Theorem (3.2.6).

If a is a properly outer automorphism of the Γ factor B, (iii) it follows that the von Neumann algebra N generated by B and $\lambda(u)$ is also a Γ factor and $\lambda(u) \in N$, again in contradiction with Theorem (3.2.6).

Corollary(3.2.9)[104]:(i) If B is a completely nonatomic finite type I von Neumann algebra then B can be embedded in $L(\mathbb{F}_\infty)$ as a maximal injective von Neumann subalgebra.

(ii) If B is a completely nonatomic type I von Neumann algebra (not necessary finite) then B can be embedded in $L(\mathbb{F}_\infty) \otimes I_\infty$ as a maximal injective von Neumann subalgebra (here $I_\infty$is the separable infinite dimensional type I factor).

Proof: Both (i) and (ii) are easy consequences of Theorem (3.2.6) and of the fact that by [121] the algebras $A_u = \{\lambda(u_n)\}'', n \geq 1$, are not unitary conjugated in $L(\mathbb{F}_\infty)$ $(u_1, u_2, \ldots,$ are the generators of $\mathbb{F}_\infty$ For instance, if $B = A_1^0 \oplus M_2(A_2^0) \oplus M_3(A_3^0) \oplus \ldots$, with A_n^0 abelian, nonatomic and $M_n(A_n^0)$ the n by n matrix algebra over A_n^0 , then take a partition of the unity $\{e_n^k\}_{n\geq k\geq 1/n\geq 1}\, in\, L(\mathbb{F}_2)$ such that $\tau(e_n^1) = \ldots = \tau(e_n^n) \neq O, n \geq 1$, andoneach projection et consider the algebra Ad $v_n^k(A_n)$, where v_n^k are partial isometries in $L(\mathbb{F}_\infty) v_n^k v_n^{k_*} = e_n^k\ and\ t) v_n^{k_*} v_n^k\ is$ a projection in A_n the same for all $k, n \geq k \geq 1$. If $B_0 = \oplus_{n,k} Ad v_n^k(A_n)$ and B_1 denotes the algebra generated in $L(\mathbb{F}_\infty)$by the normaker of ,B_0 then $B_1 \simeq B$ and B_1 is maximal injective in $L(\mathbb{F}_\infty)$

Theorem (3.2.10)[104]: Let (X, μ) be a nonatomic probability measure space and suppose $\mathbb{F}_n$ acts freely on X by measure preserving automorphisms. Denote by $M = L^\infty(X, \mu) \times \mathbb{F}_n$ the group measure algebra associated with this action and by $R_u = L^\infty(X, \mu) \times_u \mathbb{Z}$ the subalgebra of M corresponding to the action of the generator $u \in \mathbb{F}_n$on the space X. Then M and R_u are type II_1 von Neumann aIgebras and R_u , is a maximal iqjective von Neumann subalgebra of M. Moreover if u acts ergodically on X then M is a factor and $R_u$is a maximal $s\Gamma$ subfactor of M.

Proof: Denote by $A = L^\infty(X, \mu)$, so that $M = A \times \mathbb{F}_n$, $R_u = \times_u \mathbb{Z}$ since IF, acts freely on A, A is a Cartan subalgebra both in M and in R_u, [112].

The fact that M and R_u are of type II_1 follows by classical results on crossed products (see [103]).

Suppose there exists an injective von Neumann subalgebra $N \subset M$ such that $R_u \neq N$.

Then A is also a maximal abelian subalgebra in N and in fact [112] it is a Cartan subalgebra in N. In particular $N' \cap N \subset N' \cap M \subset R_u' \cap M \subset A$ so that the center of N is contained in the center of R_u which is contained in A.

Thus N is of type II_1 (since any type I central projection of N would be a central projection of type I in R_u).

We first show that $N' \cap N^{\omega} \subset R_u^{\omega}$. Suppose on the contrary that $N' \cap N^{\omega} \subset R_u^{\omega}$. As pointed, since N is an injective von Neumann algebra of type $II_1, N' \cap N^{\omega}$ is of type II,. Let $A_{\omega} = (N' \cap N^{\omega}) \cap A^{\omega} = N' \cap A^{\omega}$ and $B \subset N' \cap N^{\omega}$ a maximal abelian *subalgebra of $N' \cap N^{\omega}$ that contains A_{ω} By Lemma (3.2.1) there exist finite dimensional abelian von Neumann subalge $\tau_{\omega}(e) = 2^{-n}$for all the minimal projections $e \in A_n, n \geq 1$.

We infer that A_n is also orthogonal to A^{ω}. Indeed, since A is a Cartan subalgebra in N it follows $\mathfrak{A} = \{w$ unitary element in $N\, wA^{\omega}w^* = A^{\omega}\}$ generates N, so that by Lemma (3.2.2), $E_{A^{\omega}} \text{ o } E_{N' \cap N^{\omega}} = E_{N' \cap N^{\omega}} \text{o} E_{A^{\omega}} = E_{N' \cap N^{\omega}} = E_{A^{\omega}}$ Thus, if $x \in A_n, y \in A^{\omega}$, then $\tau_{\omega'}(xy) = \tau_{\omega}(E_{N' \cap N^{\omega}}(xy)) = \tau_{\omega'}(xE_{N' \cap N^{\omega}}(y)) = \tau_{\omega}(xE_{A^{\omega}}(y)) =$
$\tau_{\omega}(x)\, \tau_{\omega}(E_{A^{\omega}}(y)) = \tau_{\omega}(x)\, \tau_{\omega}(y)$. Now, since A_n is orthogonal to A^{ω} and $A_n \subset N' \cap N^{\omega} \subset R_u^{\omega}$, for any $g \in \mathbb{F}_n \backslash \{u^k | k \in Z)$ the algebra $\lambda(g)\, A_n \lambda(g^{-1})$ is orthogonal to $R_u^{\omega}, n \geq 1$. It follows by [106] that $\lambda(g)$ is orthogonal to $(N' \cap N^{\omega})' \cap N^{\omega}$ and thus $\lambda(g)$ is orthogonal to N for all $g \in \mathbb{F}_n \backslash \{u^k | k \in \mathbb{Z}\}$. Consequently $N\lambda(g)$ and N are mutually orthogonal linear subspaces in $L^2(M, \tau)$, in particular $A\lambda(g)$ and N are mutually orthogonal (since $A \subset N$) so that $\sum_{g \in \mathbb{F}_n \backslash \{u^k)k} L^2(A, \tau)\, \lambda(g)$ is orthogonal to N.

It follows that $N \subset L^2(R_u, t)$, hence $N \subset R_u$ which is a contradiction.

Denote by f the maximal projection in the center of N such that $Nf = R_u f$: Let $e = 1 - f$: Since $R_u \neq N, e \neq 0$. Take $x \in (N' \cap N^{\omega}) \backslash R_u^{\omega}$, It follows that $ex \in (N' \cap N^{\omega}) \backslash R_u^{\omega}$,so that we may suppose $ex = xe = x$. If $y \in N$ is an arbitrary element such that $E_{R_u}(y) = 0$ then by Lemma(3.2.4) we get

$$0 = \|y_x - x_y\|_2 \geq \left\| Y\left(x - E_{R_u^{\omega}}(x)\right)\right\|_2 h\, ence\, y \left(x - E_{R_u^{\omega}}(x)\right) = 0.$$

Moreover if $y_0, y \in N$ and $E_{R_u}(y) = 0$ then $y_0\, y(x - E_{R_u}(x)) = 0$ and $yy_0 \left(x - E_{R_u}(x)\right) = y(y_0 - E_{R_u}(y))(x - E_{R_u}(x)) + yE_{R_u}(y_0)\left(x - E_{R_u}(x)\right) = 0$ (since $E_{R_u}(yE_{R_u}(y_0)) = 0$ and $E_{R_u}(y_0 - E_{R_u}(y_0)) = 0)$. Let J be the w-closed two-sided"ideaf of N generated by all $y \in N, E_{R_u}(y) = 0$ and let p be the projection in the center of N such that $J = Np$. Since for all $y \in N$ satisfying $E_{R_u}(y) = 0$ we have eye = y it follows that $p \leq e. If\; e - p \neq 0$, there exists an element $y_0 \in N, 0 \neq y_0 = y_0(e - p)$, such that $E_{R_u}(y_0) = 0$. Indeed because otherwise $N(e - p) = R_u(e - p)$, contradicting the miximality of $f = 1 - e$. But then $y_0 \in J = Np$, which is again a contradiction. This shows that $e = p$ and by the preceding remarks for any $y \in J$ we have $y(x - y(x - E_{R_u^{\omega}}(x))$ $= 0$, in particular $e\left(x - E_{R_u^{\omega}}(x)\right) = 0$. But $e\left(x - E_{R_u^{\omega}}(x)\right) = x - E_{R_u^{\omega}}(x)$, a contradiction.

Thus R_u is maximal injective in M.

If in addition u acts ergodically on A then $\mathbb{F}_n$ acts ergodically on A so that both R_u and M are type II_1 factors. The proof that R_u is a maximal sΓ
subfactor of M is exactly the same as the proof of the maximal injectivity of R_u Indeed because from the injectivity of N we used in the preceding proof only the fact that $N' \cap N^{\omega}$ is of type II_1.

Examples (3.2.11)[104]: (i) Let (X_0, μ_0) be a probability measure space such that $L^2(X_0, \mu_0)$ has dimension at least two. For instance, consider$X_0 = \{0, 1\}$ *and* , $\mu_0(\{0\}) = \mu_0(\{1\}) = 2^{-1}$. *If* $g \in \mathbb{F}_n$ let $(X_0, \mu_0)_g = (X_0, \mu_0)$ and denote by $(X, \mu)\, \Pi_{g \in \mathbb{F}_n}, (X_0, \mu_0)$ the product probability measure space. Let σ be the Bernoulli shift action of $\mathbb{F}_n$, on X so that any $g \in \mathbb{F}_n \backslash (e)$ acts ergodically on X (in fact strongly mixing). Then $R_u = L^\infty(X, \mu) \times_{\sigma(u)} \mathbb{Z}$ is a hyperlinite subfactor of the type II_1 factor $M = L^\infty(X, \mu) \times_\sigma \mathbb{F}_n$, and by Theorem (3.2.10) R_u is a maximal $s\Gamma$ subfactor of M. Moreover by [122] and the Hilbert space lemma in [113] the action of $\mathbb{F}_n$ on (X, μ) is strongly ergodic, i.e., it has no nontrivial almost invariant sequences, so that by [107] M is a non Γ type II_1 factor (ii) Let $(X_0, \mu_0), (X, \mu) = \Pi_{g \in \mathbb{F}_n}, (X_0, \mu_0)_g$ be as in (i) and $(X_1, \mu_1) = \Pi_{k \in z}\, (X_0, \mu_0)_k$, where $(X_0, \mu_0)_k = (X_0, \mu_0)$. Let σ_1 be the following action of $\mathbb{F}_n$ on Y_1 : *if* $u, v_1, v_2, \ldots$, are as usual the generators of $\mathbb{F}_n$ then all $\sigma_1(u)\sigma_1(v_1)$ ..., act on Y, as the same Bernoulli shift over the group $\mathbb{Z}$. Consider the product action $\sigma \times \sigma_1$ of $\mathbb{F}_n$ on $X \times Y_1$, Since $\sigma(u)$ and $\sigma_1(u)$ are strongly mixing, $(\sigma \times \sigma_1)(u)$ is ergodic, so that the corresponding group measure algebra $M = L^\infty(X \times Y_1, \mu \times \mu_1,) X_{\sigma \times \sigma_1}\, \mathbb{F}_n$, is a type$\mathrm{II}_1$, factor and $R_u = L^\infty(X \times Y_1, \mu \times \mu_1) X_{\sigma \times \sigma_1(u)}\, \mathbb{Z}$ is a hyperfinite subfactor of M. By Theorem (3.2.10) R_u is a maximal $s\Gamma$ subfactor of M. Since $\sigma \times \sigma_1$ has nontrivial almost invariant sequences (because the action $\sigma_1$is amenable), M is a Γ factor and since $R_u \neq M, M$ *is* $w\Gamma$ (because if M would be $s\Gamma$ then maximality of R_u would be contradicted). The fact that the Γ factor M is $w\Gamma$ follows also by [106].
(iii) $(X_0, \mu_0), (X, \mu) = \Pi_{g \in \mathbb{F}_n}, (X_0, \mu_0)_g$ be as in (i), $\mathbb{F}_{n-1}$ the subgroup of $\mathbb{F}_n$ generated by $v_1, u_2, \ldots$, and $(Y_2, \mu_2) == \Pi_{g \in \mathbb{F}_{n-1}}\, (X_0, \mu_0)_g$.

Consider the action σ_2 of F_2, on Y_2 as follows : $\sigma_2(u)$acts trivially on Y_2 and $\sigma_{2|F_{n-1}}$is the Bernoulli shift over the group F_{n-1} Then the action $\sigma \times \sigma_2$ of$\mathbb{F}_n$*on* $(X \times Y_2, \mu \times \mu_2)$ is free, ergodic, but the action of $\sigma \times \sigma_2(u)$ is not ergodic, it acts trivially on sets of the form $X \times A, A \subset Y_2$. Thus in this case $M = L^\infty(X \times Y_2, \mu \times \mu_2) x_{\sigma \times \sigma_2}\, \mathbb{F}_n$ is a type II_1 factor and $R_u \simeq R \otimes A_0$where A_0 is a completely nonatomic abelian von Neumann algebra. By the same arguments as in (i) and (ii) we get that for $n = 2, M$ is a $w\Gamma$ factor (because σ_2 is amenable) and for $n \geq 3$ M is non Γ (because in this case σ_2 is strongly ergodic).
(iv) Let $x_0 = \{0, 1\}$ with the measure $\mu_0(\{O\}) = \mu_0(\{I\}) = 2^{-1}$ and let $z_1, z_2, \ldots$, be a Bore1 partition of $(Y_1, \mu_2) = \Pi_{k \in z}\, (X_0, \mu_0)_k$ with $\mu(Z_k) = 2^{-k}, k \geq 1$. Then $\left(Z_k, \mu_{I|Z_k}\right) \simeq (Y_1, 2^{-k}\mu_1)$. Now let σ_0 be the following action of $\mathbb{F}_n$*on* Y_1 : $\sigma_0(u)$ is trivial on Z_1 and it is the Bernoulli shift on $Z_k, k \geq 2$, via the isomorphism$\left(Z_k, \mu_{I|Z_k}\right) \simeq (Y_1, 2^{-k}\mu_1)_{\sigma_{0|\mathbb{F}_{n-1}}}$ acts on Y_1 as the Bernoulli shift via the isomorphism $(Y_1, \mu_1) \simeq \Pi_{g \in \mathbb{F}_n}, (X_0, \mu_0)_g$ Then take the product action $\sigma \times \sigma_0$, *of* $\mathbb{F}_n$, *on* $X \times Y_1$ Since σ *and* σ_0 are strongly mixing the corresponding M is a type II_1 factor. The subalgebra R_u is isomorphic in this case to $R \otimes A_0$ where A_0 is abelian of the form $A_0 = A_1 \otimes A_2$with A_1 completely nonatomic and A_2 atomic and infinite dimensional. Again by113]], [107], if $n = 2$ then M is $\mathrm{w}\Gamma$ and if $n \geq 3$ then Misnon Γ.
Note that by obvious modifications of this example we can choose the action $\sigma_0$such that the abelian algebra A_0 (the center of R,) is of any form we like.

(v) Let $\mathbb{F}_2^0$ be the subgroup of $SL(2,\mathbb{Z})$ generated by $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ so that $\mathbb{F}_2^0$ is isomorphic to the free group $\mathbb{F}_2. Let\ \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus with the normalized Haar measure $\mu. Let\ SL(2,\mathbb{Z})$ act on $\mathbb{T}^2$ as the group of linear automorphisms and denote by $\sigma'$the restriction of this action to $\mathbb{F}_2^0$ $As\ \sigma'$is well known to be ergodic, the algebra $M = L^\infty(\mathbb{T}^2,\mu) \times_{\sigma'} \mathbb{F}_2$, is a type II_1 factor. If R_u is the von Neumann subalgebra of M generated by the action of the element $u = (\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}) \in \mathbb{F}_2^0$ then R_u has diffuse center but it is maximal injective in M by Theorem (3,2.10). We mention that it is not known whether the action of $\mathbb{F}_2^0$ on $\mathbb{T}^2$ is strongly ergodic, although the global action of $SL(2,\mathbb{Z})$ was showd to be strongly ergodic in [122].

Let us summarise the conclusions of the preceding examples, using then notations of Theorem (3.2.10):

Proposition (3.2.12)[104]: Let A_0 be an arbitrary separable abelian von Neumann algebra. There exist free ergodic measure preserving actions of $\mathbb{F}_n, n \geq 2$, on a nonatomic probability measure space (X,μ) such that $R_u \simeq R \otimes A_0$, and such that M is a non Γ or a w Γ factor of type II_1.

Note that Theorem (3.2.10) and Examples (3.2.11) also provide examples of maximal amenable subequivalence relations of the measured equivalence relation $R_{\mathbb{F}_n}$ implemented on (X,μ) by the action of the group $\mathbb{F}_n$.

We mention now a consequence of [92]:

Theorem(3.2.13)[104]: If M is a separable type II_1 factor then M contains the hyperfinite factor R as a maximal injective von Neumann subalgebra.

Proof: By [92] there exists a hyperfinite subfactor $R_0 \subset M$ such that
$R_0' \cap M = \mathbb{C}$. So, if R is a maximal injective von Neumann subalgebra that contains R_0 then $R' \cap M \subset R_0' \cap M = \mathbb{C}$ and thus R is a factor. By [96] R is the hyperfinite type II_1 factor.

We close with two problems. The first one, if answered in the affirmative, would considerably enlarge our class of examples. The second one is related to the proof of Theorem (3.2.10), but also has an independent interest.

Problems(3.2.14)[104]: If M_1, M_2 are type II_1 factors and $B_1 \subset M_1, B_2 \subset M_2$ are maximal injective von Neumann algebras, is $B_1 \otimes B_2$maximal injective in $M_1 \otimes M_2$? Is this true at least for $M_2 = B_2 = R$?

Let $R_I \subset R$ be a hyperlinite subfactor such that $R_0' \cap R = \mathbb{C}$ and $R' \cap R^\omega \subset R_1^\omega$ for some free ultrafilter $\omega\ on\ \mathbb{N}$. Does it follow that $R_1 = R$?

Let (X,μ) be a nonatomic probability measure space and suppose $\mathbb{F}_2$ acts freely on $(X,\mu)$by measure preserving automorphisms. Let u, v be the generators of $\mathbb{F}_2$. denote $A = L^\infty(X,\mu), M = A \times \mathbb{F}_2, R_u = A \times_u \mathbb{Z}$ and by $\lambda(\mathrm{g}), \mathrm{g} \in \mathbb{F}_2$ the unitaries of M canonically implementing the action of $\mathbb{F}_2$ on A. Suppose in addition that both u and v act ergodically on A and that there exists an automorphism $\Theta\ on\ M$ such that $\Theta(\lambda(u)) = \lambda(v), \Theta(A) = A$. For examples of such a situation see Examples (3.2.11), (i) and (ii); so, R, is hyperfinite and by Theorem (3.2.10) it is a maximal sr subfactor of M.

Denote by N the algebra of 2 by 2 matrices over M and $R = \{x \oplus \Theta(x) | x \in R_u\} \subset N$. Thus R is isomorphic to R_u and in fact, if $e = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \in M_2(M) = N$, then $Re = R_u$ Note that if $\mathbb{F}_2$ acts as in Examples (3.2.11), (i) then M and N are non Γ, if $\mathbb{F}_2$ acts as in 2º then M and N are $w\Gamma$.

Theorem (3.2.15)[104]: With the above notations, R is a maximal $s\Gamma$ subfactor in N. In particular it is a maximal hyperfinite subfactor in N, but $R' \cap N = Ce + C(1-e)$.

Proof: We shall assume on the contrary that there exists an $s\Gamma$ subfactor N_0 in N such that $R \subset N_0, R \neq N_0$ First of all note that N_0 has elements of the form $x = \begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix}$ $with\, x^{12} \neq 0$. Indeed, because otherwise $e \in N_0' \cap N\ and\ so\ N_0 e \simeq N_0$, would be an $s\Gamma$ subfactor of M. By Theorem (3.2.10), $R_u = N_0 e\ and\ thus\ R = N_0$ which contradicts our assumption.

Next we show that R has a central sequence $(x_n)_n$ with

$$x_n = \begin{pmatrix} x_n^{11} & x_n^{12} \\ x_n^{21} & x_n^{22} \end{pmatrix},$$

such that $\|x_n^{12}\|_2 \geq \delta > 0$, for all $n \geq 1$. We do this in the following two lemmas. In the third lemma we show that in fact there also exists an element

$$y = \begin{pmatrix} y^{11} & y^{12} \\ y^{21} & y^{22} \end{pmatrix} \in N_0,$$

such that $\|y^{21} x_n^{12}\|_2 \geq c > 0, n \geq 1$.

Lemma(3.2.16)[104]: $eN_0 e \not\subset R_u$.

Proof: Let $x = x^* \in N_0 with\, x^{l2} \neq 0$. Then the element $x(\lambda(u^n) \oplus \Theta(\lambda(u^n)))\, x^* = x(\lambda(u^n) \oplus \lambda(v^n))\, x^*$ belongs to N_0 and

$$\mathrm{x}(\lambda(\mathrm{u}^n) \oplus \lambda(\mathrm{v}^n))\mathrm{x}^* = \begin{pmatrix} x^{11} & x^{12} \\ x^{12^*} & x^{22} \end{pmatrix} \begin{pmatrix} \lambda(u^n) & 0 \\ 0 & \lambda(v^n) \end{pmatrix} \begin{pmatrix} x^{11} & x^{12} \\ x^{12^*} & x^{22} \end{pmatrix}$$
$$= \begin{pmatrix} x^{11}\lambda(u^n)x^{11} + x^{12}\lambda(v^n)x^{12}\ ) & * \\ * & * \end{pmatrix}$$

If $eN_O e \subset R_u$ then $x^{11} \in R$ and $x^{11}\lambda(u^n)\, x^{11} + x^{12}\lambda(v^n) x^{12^*} \in R_u$
Since $\lambda(u^n) \in R_u$ we get $x^{12}\lambda(v^n)x^{12^*} \in R_u$ for all $n \in \mathbb{Z}$. Thus
$yx^{12}\lambda(v^n)x^{12^*}y^*$ is in R_u for all $y \in R_u, n \in \mathbb{Z}$. As $\lambda(g)R_u\lambda(g^{-1})$ and
$A_v = \{\lambda(v)\}''$ are mutually orthogonal subalgebras in M for all
$\mathrm{g} \in \mathbb{F}_2$, it follows that $\lambda(g)yx^{12}A_v x^{12^*}y^*\lambda(\mathrm{q}^{-1})$ and $A_v \ominus \mathbb{C}$ are
mutually orthogonal linear subspaces in $L^2(M,\tau), \mathrm{g} \in \mathbb{F}_2$ So, if
$b_1, b_2 \in A_v$ then $\tau((b_2 - t(b_2))\, \lambda(\mathrm{g})yx^{12}b_1 x^{12^*}y^*\lambda(\mathrm{g}^{-1}) = 0$, or equivalent$\tau(b_2\lambda(\mathrm{g})yx^{12}b_1x^{12^*}y^*\lambda(\mathrm{g}^{-1}) = \tau(b_2)(\lambda(\mathrm{g})yx^{12}b_1x^{12^*}y^*\lambda(\mathrm{g}^{-1}))$.

In particular if we take an arbitrary $\varepsilon > 0$ and a partition of the unity $e_1, e_2, \dots, e_m in\ A_v such\ that\ \tau(e_i) < \varepsilon,\ m \geq i \geq 1$ then we have $\tau(\lambda(\mathrm{g})yx^{12}e_i x^{12^*}y^*\lambda(\mathrm{g}^{-1})) < \varepsilon\|x^{12}\|^2\|y\|^2$ so that if A_0 denotes the algebra generated by $e_1 \dots, e_m$we get

$$|\tau(\lambda(\mathrm{g})yx^{12}|^2 \leq \left\|E_{A_0' \cap M}(\lambda(\mathrm{g})yx^{12})\right\|_2^2$$
$$= \left\|\sum_i e_i\lambda(\mathrm{g})yx^{12}e_i\right\|_2^2 = \sum_i \|e_i\lambda(\mathrm{g})yx^{12}e_i\|_2^2$$

$$= \sum_i \tau(e_i\lambda(\mathrm{g})yx^{12}e_ix^{12^*}y^*\lambda(\mathrm{g}^{-1}))$$
$$< \varepsilon\|x^{12}\|^2\|y\|^2\sum_i \tau(e_i) = \varepsilon\|x^{12}\|^2\|y\|^2$$

Thus $t(\lambda(\mathrm{g})yx^{12}) = 0$ for all $\mathrm{g} \in \mathbb{F}_2, y \in R_u$, and if we take $y \in A$ we obtain that x^{12} is orthogonal to $A\lambda(\mathrm{g})$ in $L^2(M,\tau)$ for all $\mathrm{g} \in \mathbb{F}_2$ so that x^{12} is orthogonal to M. This is a contradiction.

Lemma(3.2.17)[104]: N_0 has a cent and sequence $(x_n)_n$ with $\|x_n^{12}\|_2 \geq \delta > 0, n > 1$.

Proof: If we assume the contrary, then e commutes with $N_0' \cap N_0^\omega$. Since $N_0' \cap N_0^\omega$ is a type II_1 von Neumann algebra it follows that $(N_0' \cap N_0^\omega.)$, is also of type II_1 Let M_1 be the von Neumann algebra generated in M by eN_0e. Then $R_u \subset M_1, M_1$is a factor (because $M_1' \cap M_1 \subset M_1' \cap M \subset R_u' \cap M = \mathbb{C}$) and by the preceding lemma $R_u \neq M_1$ as $M_1' \cap M_1^\omega$containse$(N_0' \cap N_0^\omega)$e it follows that $M_1' \cap M_1^\omega$ is noncommutative, so that M_1 is an $s\Gamma$ factor, contradicting Theorem (3.2.10).

Lemma(3.2.18)[104]: There exists an element $y \in N_0$, and a central sequence $(x_n)_n$ in N_0 such that $\|y^{21}x_n^{12}\|_2 \geq c > 0, n \geq 1$.

Proof: By Lemma (3.2.17) there exists a central sequence $(y_n)_n$ such that $\|y_n\| \leq 1, \|x_n^{12}\|_2 \geq \delta > 0, n > 1$. We claim that there exist $c > 0, n_0 \in \mathbb{N}$ and a subsequence $\left(y_{n_k}^{12}\right)_k of\ (y_n^{12})_n$ such that $\left\|y_{n_0}^{12^*}y_{n_k}^{12}\right\|_2 \geq c > 0$ for all $k \geq 1$. Denote $\tilde{y}_0 = (y_n^{12})_n \in N_0^\omega$ and assume on the contrary that for any $k \geq 1$ there exists n_k $(with\, n_k > n_{k-1}$ such that $\left\|x_k^{12^*}x_m^{12}\right\|_2 < 2^{-k}$ for all $m \geq n_k$ Thus $\tilde{y}_1 = (y_n^{12})_n \in N_0^\omega$ satisfies $\tilde{y}_0^*\tilde{y}_1 = 0\ in\ N_0^\omega$ Then construct $\tilde{y}_2$ as a subsequence of $\tilde{y}_1$ such that $\tilde{y}_1^*\tilde{y}_2 = 0. \tilde{y}_2$ will also satisfy $\tilde{y}_1^*\tilde{y}_2 = 0$. Recursively we get n + 1 elements $\tilde{y}_0, \tilde{y}_1, \dots, \tilde{y}_n$ in N_0^ω with$\tilde{y}_l^*\tilde{y}_j = 0\ if\ i \neq j, \|\tilde{y}_l\| \leq 1$ and $\|\tilde{y}_l\|_2 \geq \delta$. This is a contradiction if $n > \delta^{-1}$.

End of the proof of Theorem (3.2.15). By Lemma (3.2.18) there exists a sequence $(x_n)_n$in N and $y \in N$ such that $\|x_n\| \leq 1, n \geq 1, \|[x_n, \lambda(u)\oplus\lambda(v)]\|_2 \to 0, \|x_n, y\|_2 \to 0$ and $\|y^{21}x_n^{12}\|_2 \geq c > 0, n \geq 1$ It follows that:

$$\left\|\lambda(u^k)x_n^{11}\lambda(u^{-k}) - x_n^{11}\right\|_2 \to 0\left\|\lambda(v^k)x_n^{22}\lambda(v^{-k}) - x_n^{22}\right\|_2 \to 0$$
$$\left\|\lambda(v^k)x_n^{21}\lambda(u^{-k}) - x_n^{21}\right\|_2 \to 0\left\|\lambda(u^k)x_n^{12}\lambda(v^{-k}) - x_n^{12}\right\|_2 \to 0 \text{ for all } k \in \mathbb{Z} \tag{1}$$
$$\|y^{22}x_n^{22} + y^{21}x_n^{12} - x_n^{22}y^{22} - x_n^{21}y^{12}\|_2 \to_n 0. \tag{2}$$

We shall use from now on. Similar computations as in the proof of Lemma (3.2.4) will show that the element $y^{21}x_n^{12}$ makes it impossible for (2) to hold. So let $\varepsilon > 0$ by the Kaplansky density theorem there exist $y_0^\cup \in \mathbb{F}$ (polynomials in $\mathbb{F}_2$with coefficients in A) such that $\|y^\cup - y_0^\cup\|_2 < \varepsilon, \|y_0^\cup\| \leq \|y^\cup\|. Let\ N_0 - 1$ be the maximal length of a word appearing in the supports of $y_0^\cup, 1 \leq i,j \leq 2. Let\ N_1$ be a multiple of $4N_0$, such that $N_1 > 3\varepsilon^{-2} N_0. Let\ n_1 = n_1(\varepsilon, N_1)$ be such that for $n \geq n_1$ we have:

$$\left\|\lambda(u^k)x_n^{21}\lambda(u^{-k}) - x_n^{11}\right\|_2 < \varepsilon, \left\|\lambda(v^k)x_n^{22}\lambda(v^{-k}) - x_n^{22}\right\|_2 < \varepsilon,$$
$$\left\|\lambda(u^k)x_n^{12}\lambda(v^{-k}) - x_n^{12}\right\|_2 < \varepsilon,$$
$$\left\|\lambda(v^k)x_n^{21}\lambda(u^{-k}) - x_n^{21}\right\|_2 < \varepsilon \text{ for } |k| \leq N_1 \tag{3}$$
$$lim\ sup_n\|y_0^{22}x_n^{22} + y_0^{21}x_n^{12} - x_n^{22}y_0^{22} - x_n^{21}y_0^{12}\|_2 \leq \varepsilon \tag{4}$$

Note that we also have $\|y_0^{21}x_n^{12}\|_2 \geq c - \varepsilon$.

Let $T_0^1 = \{g \in\vdots\ \mathbb{F}_2\ | g$ begins with a power of u not larger in absolute value than $2N_0 - 1\}, T_0^1 = \{g \in\vdots\ \mathbb{F}_2\ | g$ ends with a power of v not larger in absolute value than $2N_0 - 1\}, T_0 = T_0^1 \cup T_0^2$ We show first that $\left\|(x_n^{12})_{T_0}\right\|_2 < 3\varepsilon$. 'Indeed, we have that $\{u^{4N_0k}T_0^1 v^{4N_0k}\}_{k\in z}$ are disjoint sets and for $4N_0|k| \leq N_1$, we get

$$\left\|\lambda(u^{4N_0k})(x_n^{12})_{T_0^1}\lambda(v^{4N_0k}) - (x_n^{12})_{u^{4N_0k}T_0^{1-4N_0k}}\right\|_2$$

$$= \left\|\lambda(u^{4N_0k})x_n^{12}\lambda(v^{4N_0k}) - x_n^{12})_{u^{4N_0k}T_0^{1-4N_0k}}\right\|_2$$

$$\leq \left\|\lambda(u^{4N_0k})x_n^{12}\lambda(v^{4N_0k}) - x_n^{12}\right\|_2 < \varepsilon$$

Using the parallelogram identity and summing up over all $|k| \leq (4N_0)^{-1}N_1, k \neq 0,$ we get

$$2(4N_0)^{-1}N_1\left\|(x_n^{12})_{T_0^1}\right\|_2^2$$

$$\leq 2\sum_k \left\|\lambda(u^{4N_0k})(x_n^{12})_{T_0^1}\lambda(v^{-4N_0k}) - (x_n^{12})_{u^{4N_0k}T_0^1v^{-4N_0k}}\right\|_2^2$$

$$+ 2\left\|\sum_k (x_n^{12})_{u^{4N_0k}T_0^{1-4N_0k}}\right\|_2^2 \leq 2(4N_0)^{-1}N_1\varepsilon^2 + 2\|x_n^{12}\|_2^2$$

so that $\left\|(x_n^{12})_{T_0^1}\right\|_2^2 < \varepsilon^2 + \varepsilon^2$; similarly $\left\|(x_n^{12})_{T_0^1}\right\|_2^2 < 2\varepsilon^2$ and thus

$$\left\|(x_n^{12})_{T_0}\right\|_2 \leq \left\|(x_n^{12})_{T_0^1}\right\|_2 + \left\|(x_n^{12})_{T_0^2}\right\|_2 < 3\varepsilon$$

Let $T_1 = \mathbb{F}_2 \backslash T_0$ and $T = U\{\, gT_1\ | g \in \mathbb{F}_2$ has a length not larger than $N_0 - 1\}$. Clearly $y_0^{21}x_n^{12} = y_n^{21}(x_n^{12})_{T_1} + x_0^{21}(x_n^{12})_{T_0}$ and $y_0^{21}(x_n^{12})_{T_1} = \left(y_0^{21}(x_n^{12})_{T_1}\right)_T$, so that

$$\|(y_0^{21}x_n^{12})_T\|_2 \geq \left\|y_0^{21}(x_n^{12})_{T_1}\right\|_2 - \left\|y_0^{21}(x_n^{12})_{T_0}\right\|_2$$

$$\geq \|y_0^{21}x_n^{12}\|_2 - 2\left\|y_0^{21}(x_n^{12})_{T_0}\right\|_2 \geq \|y_0^{21}x_n^{12}\|_2 - 6\varepsilon\|y^{21}\|$$

In particular we have $\|y_0^{21}x_n^{12}\|_T \geq c - \varepsilon - 6\varepsilon\,\|y^{21}\|$. To get a contradiction from this inequality and (4) it will be sufficient to show that $y_0^{22}x_n^{22}, x_n^{22}y_0^{22}, x_0^{21}y_0^{12}$ have small norms on the set T. This is easy to see for $x_0^{21}y_0^{12}$, since by the same computation as for $y_0^{21}x_n^{12}$ its norm is concentrated on T^{-1} and $T \cap T^{-1} = \phi$.

The other two elements in (4) can be treated in the same way, so let us do it for $\|(y_0^{22}x_n^{22})_T\|_2$. Denote as in Lemma (3.2.4) $, S, = \{v^k | k \in \mathbb{Z}\}, S_1 = \{g \in \mathbb{F}_2 | g$ begins and ends with powers of v greater than $2N_0 - 1$ in modulus$\}$, $S_0 = (\mathbb{F}_2\backslash S_v)\backslash S_1$. As in the proof of Lemma (3.2.4), we may suppose that n_1 is such that $\left\|(x_n^{22})_{S_0}\right\|_2 3\varepsilon$, for all $n \geq n_1$ For any $g \in \mathbb{F}_2$, of length not larger than $N_0 - 1$, we have $gS_1 \cap T = \phi, gS_v \cap T = \phi$. Indeed because in the first $2N_0$ letters of a word in T there are more u's than u's, while a word in gS_1 or in gS_v is in the opposite situation, and also because any word in T has more than $3N_0$ letters, with some nonzero power of v at the end. It follows that

$$\|(y_0^{22}x_n^{22})_T\|_2 \leq \left\|y_0^{22}(x_n^{22})_{S_0}\right\|_2 \leq \|y^{22}\|.\left\|(x_n^{22})_{S_0}\right\|_2.$$

We have thus obtained that for $n \geq n_1$,

$$\|y_0^{22}x_n^{22}+y_0^{21}x_n^{12}-x_n^{22}y_0^{22}-x_n^{21}y_0^{12}\|_2$$
$$\geq \|(y_0^{22}x_n^{22})_T+(y_0^{21}x_n^{12})_T-(x_n^{22}y_0^{22})_T-(x_n^{21}y_0^{12})_T\|_2$$
$$\geq \|(y_0^{21}x_n^{12})_T\|_2-\|(y_0^{22}x_n^{22})_T\|_2-\|(x_n^{22}y_0^{22})_T\|_2-\|(x_n^{21}y_0^{12})_T\|_2$$
$$\geq c-\varepsilon-6\varepsilon\|y^{21}\|-3\varepsilon\|y^{22}\|-3\varepsilon\|y^{22}\|-3\varepsilon\|y^{12}\|$$
So, if ε is small enough this is in contradiction with (4). Hence our initial assumption on the existence of an $s\Gamma$ subfactor N_0 of N such that $R\subset N_0, R\neq N_0$ lead to a contradiction. It follows that R is a maximal $s\Gamma$ subfactor in N.

Examples (3.2.19)[104]: (i) A maximal hyperfinite subfactor -with noncommutative relative commutant can be constructed as follows: Let' Θ, R_u, M be as in Theorem (3.2.15) and denote $N = M_3(M)$ the 3 by 3 matrix algebra over $M, R = \{x\oplus x\oplus \Theta(x)|x\in R_u\}$. Then R is maximal hyperfiite in N (in fact it is maximal $s\Gamma$) and $R'\cap N \simeq M_2(\mathbb{C})\oplus\mathbb{C}$.

(ii) A more general -example than Theorem (3.2.15), is the following: Let $\mathbb{F}_n$be the free group with n generators, that we denote by' $u_1, u_n, \dots, u_n$ $(\infty > n\geq 2)$ and suppose $\mathbb{F}_n$ acts freely and ,ergodically by measure preserving transformations on the nonatomic probability measure space (X,μ). As in Theorem (3.2.15), denote $A = L^\infty(X,\mu), M = A\times F_n, R_1 = A\times_{u_1}\mathbb{Z}$and by $\lambda(\mathrm{g}), \mathrm{g}\in\mathbb{F}_n$ the unitaries in M canonically implementing the action of $\mathbb{F}_n\ on\ A$. Suppose there exists an automorphism $\Theta\in Aut(M)$ such that $\Theta(A) = A, \Theta(u_1) = u_{i+1} n-1\geq i\geq 1$ and let $N = M_n(M), R = \{x\oplus\Theta(x)\oplus\dots\oplus\Theta^{n-1}(x)|x\in R_1\}$.

Then R is a maximal hyperiinite subfactor of N (in fact a maximal $s\Gamma$ subfactor) and $R'\cap N\simeq C^n$.

In both examples (i) and (ii) it follows by Examples (3.2.11) , (i), (ii) that M and N can be chosen either non Γ or $w\Gamma$.

The proofs of Examples (3.2.19), (i) and (ii), aside from some ‘obvious modifications, follow step by step the proof of Theorem (3.2.15). So we have in conclusion:

Theorem (3.2.20)[104]: (i) For any $n\geq 2$ there exist type II_1wΓ and non Γfactors M with maximal hypeflnite subfactors R such that $R'\cap M\simeq\mathbb{C}^1$.

(ii) There exist II_1wΓ and non Γ factors with maximal hypflnite subfactors having noncommutative reliitive cornmutant.

The above theorem and Theorem (3.2.13) show that a first invariant to consider for the classification (up to conjugation by automorphisms) of the maximal hyper- finite subfactors of a type II_1 factor M is the type of their relative cornmutant in M.

Corollary (3.2.21)[260]: Let B_r, B_{r+1} be von Neumann subalgebras of M and suppose that the group $u = \{w_{r-2}$ unitary in $B_{r-2}w_{r-2}[B_{r+1}w_{r-2}^* = B_{r+1}]$ generates B_r then $E_{B'_{r+1}}\circ E_{B'_r\cap M} = E_{B'_r\cap M}\circ E_{B_{r+1}} = E_{B'_r\cap M_2}$.

Proof: For $x\in M$, let $K_X = \overline{co}^{w_{r-2}}\{uxu^*|u$ unitary in $U\}$. Then K_x is a convex weakly compact subset of M and by the inferior semicontinuity of the application $x\to t\ \|x\|_2$ it follows that there exists $E(x)\in K_x$, such that $\|E(x)\|_2 = \inf\{\|y\|_2\ y\in K_x\}$. Since $\|\ \|_2$is a Hilbert norm and K_x is convex it follows that $E(x)$is the unique element in K_x, with this property. Moreover, since $\mathcal{U}_{r-2}$ is a group, $w_{r-2}E(x)w_{r-2}^*\in K_x$ for all w_{r-2} in $\mathcal{U}_{r-2}$ and $\|w_{r-2}E(x)w_{r-2}^*\|_2 = \|E(x)\|_2$ so that $w_{r-2}E(x)w_{r-2}^* = E(x)$. Consequently $E(x)\in\mathcal{U}'_{r-2}\cap M = B'_r\cap M$ and E is a well-defined function from M to $B'_r\cap M$. If $x\in B'_r\cap M$

then clearly $K_x = \{x\}$ so that $E(x) = x$. If $x \in M$ is orthogonal to $B_r' \cap M$ (as an element in $L^2(M,\tau)$) then the set K_x is orthogonal to $B_r' \cap M$ (since $w_{r-2}xw_{r-2}^*$ is orthogonal to $B_r' \cap M$ for all unitaries $w_{r-2} \in \mathcal{U}_{r-2}$). This means that $E(x) = 0$. It follows that E(x) is the orthogonal projection of x onto $B_r' \cap M$ that is, $E(x) = E_{B_r' \cap M}(x)$.

Now, for $x \in B_{r+1}$ we get $w_{r-2}xw_{r-2}^* \in B_{r+1}$ for all $w_{r-2} \in \mathcal{U}_{r-2}$ so that $K_x \subset B_{r+x-1}$, and thus $E_{B_r' \cap M}(x) \in B_{r+1}$. Since we also have $E_{B_r' \cap M}(x) \in B_r' \cap M$ we get $E_{B_r' \cap M}(B_{r+1}') \subset B_r' \cap B_{r+1}$. So, if p and q denote the extensions of $E_{B_r' \cap M}$ and, respectively, $E_{B_{r+1}}$ to $L^2(M,\tau)$ then the left suppot of pq is equal to $p \wedge q$. It follows that $pq = p \wedge q = qp$.

Corollary (3.2.22)[260]: N_{r-1} has a cent and sequence $(x_{1+\epsilon})_{1+\epsilon}$ with $\|x_{1+\epsilon}^{12}\|_2 \geq \delta > 0$, $\epsilon > 0$.

Proof: If we assume the contrary, then e commutes with $N_{r-1}' \cap N_{r-1}^{\omega}$. Since $N_{r-1}' \cap N_{r-1}^{\omega}$ is a type II_1 von Neumann algebra it follows that ($N_{r-1}' \cap N_{r-1}^{\omega}$.), is also of type II_1 Let M_r be the von Neumann algebra generated in M_{r-2} by $eN_{r-1}e$. Then $R_{u_{r-2}} \subset M_r, M_r$ is a factor (because $M_r' \cap M_r \subset M_r' \cap M_{r-2} \subset R_{u_{r-2}}' \cap M_{r-2} = \mathbb{C}$) and by the preceding lemma $R_{u_{r-2}} \neq M_r$ as $M_r' \cap M_r^{\omega}$ contains $e(N_{r-1}' \cap N_{r-1}^{\omega})$ e it follows that $M_r' \cap M_r^{\omega}$ is noncommutative, so that M_r is an $s\Gamma$ factor, contradicting Theorem (3.2.10).

Chapter 4
Quasi-regular and Induced Representations of the Infinite-Dimensional Nilpotent Group

We show that construction uses the infinite tensor product of arbitrary Gaussian measures in the spaces $\mathbb{R}^m$ with $m > 1$ extending in a rather subtle way for the infinite tensor product of one-dimensional Gaussian measures. It depends on two completions $\widetilde{\mathrm{H}}$ and $\widetilde{\mathrm{G}}$ of the subgroup H and the group G, on an extension $\tilde{\mathrm{S}} : \widetilde{\mathrm{H}} \to \mathrm{U(V)}$ of the representation S : H → U(V) and on a choice of the G-quasi-invariant measure μ on an appropriate completion $\widetilde{\mathrm{X}} = \widetilde{\mathrm{H}} \backslash \widetilde{\mathrm{G}}$ of the space H\G. We consider the "nilpotent" group $\mathrm{B}_0^{\mathbb{Z}}$ of infinite in both directions upper triangular matrices and the induced representation corresponding to the so-called generic orbits.

Section (4.1): Infinite-Dimensional Nilpotent Group

For (X, B) be a measurable space and let Aut*(X)* denote the group of all measurable automorphisms of the space *X*. With any measurable action $\alpha : G \to$ Aut*(X)* of a group* *G* on the space *X* and a *G*-quasi-invariant measure μ *on* X one can associate a unitary representation $\pi^{\alpha,\mu,X} : G \to U(L^2(X,\mu))$, of the group G by the formula $(\pi_t^{\alpha,\mu,X} f)(x) = (d\mu(\alpha_{t^{-1}}(x))/d\mu(x))^{1/2} f(\alpha_{t^{-1}}(x)), f \in L^2(X,\mu)$. Let us set $\alpha(G) = \{\alpha_t \in Aut(X) \mid t \in G\}$. Let $\alpha(G)'$ be the centralizer of the subgroup $\alpha(G)$ in $Aut(X)$: $\alpha(G)' = \{g \in Aut(X) \mid \{g, \alpha_t\} = g\alpha_t g^{-1}\alpha_t^{-1} = e\ \forall t \in G\}$. The following conjecture has been discussed in [146]–[148].

Conjecture (4.1.1)[123]: The representation $\pi^{\alpha,\mu,X} : G \to U(L^2(X,\mu))$ is irreducible if and only if :

(i) $\mu^g \perp \mu\ \forall g \in \alpha(G)' \setminus \{e\}$ (where ⊥ stands for singular),

(ii) the measure μ *is G*-ergodic.

We recall that a measure μ is *G*-ergodic if $f(\alpha_t(x)) = f(x)\ \forall t \in G$ implies $f(x) = const\ \mu$ a.e. for all functions $f \in L^1(X,\mu)$.

We shall show Conjecture (4.1.1) in the case where *G* is the infinite-dimensional nilpotent group $G = B_0^{\mathbb{N}}$ of finite upper-triangular matrices of infinite order with unities on the diagonal, the space $X = X^m$ being the set of left cosets $G_m \setminus B^{\mathbb{N}}, G_m$ being suitable subgroups of the group $B^{\mathbb{N}}$ of all upper-triangular matrices of infinite order with unities on the diagonal, and μ an infinite tensor product of Gaussian measures on the spaces $\mathbb{R}^m$ with some fixed $m > 1$.

A more detailed explanation of the concepts used here is given in the following.

Let *G* be a locally compact group. The *right* ρ $(respectively\ left\ \lambda)$ *regular representation* of the group *G* is a particular case of the representation $\pi^{\alpha,\mu,X}$ with the space *X* = *G*, the action α being the right action $\alpha = R$ (respectively the left action $\alpha = L$), and the measure μ being the right invariant Haar measure on the group *G* (see, [131], [139], [140], [160]).

A quasiregular representation of a locally compact group *G* is also a particular case of the representation $\pi^{\alpha,\mu,X}$ (see, for example, [160]) with the space *X* = *H* \ *G*, where *H* is a subgroup of the group *G*, the action α being the right action of the group *G* on the space *X* and the measure μ being some quasi-invariant measure on the space *X* (this measure is unique up to a scalar multiple). We remark that in [139], [140] this representation has also been called *geometric representation*.

We will consider the approach which deals with analogs for infinite dimensional groups of the regular and quasiregular representations of finite-dimensional groups.

Let *G* be an infinite-dimensional topological group. To define an analog of the regular representation, let us consider some topological group $\tilde{G}$, containing the initial group *G* as a dense subgroup, i.e. $\bar{G} = \tilde{G}$ ($\bar{G}$ being the closure of *G*). Suppose we have some quasi-invariant measure μ on $X = \tilde{G}$ with respect to the right action of the group $G, i.e. \alpha = R, R_t(x) = xt^{-1}$. In this case we shall call the representation $\pi^{\alpha,\mu,\tilde{G}}$ an analog of the regular representation. We shall denote this representation by $T^{R,\mu,}$ and the Conjecture (4.1.1) is reduced to the following Ismagilov conjecture.

Conjecture(4.1.2)[123]:(Ismagilov,1985) The right regular representation $T^{R,\mu} : G \to U(L^2(\tilde{G},\mu))$ is irreducible if and only if :

(i) $\mu \quad \perp \mu \forall t \in G\setminus\{e\}$,

(ii) the measure μ is *G*-ergodic.

The work [145] initiated the study of representations of current groups, i.e. groups $C(X,U)$ of continuous mappings $X \to U$, where X is a finite-dimensional Riemannian manifold and *U* is a finite-dimensional Lie group.

The regular representation of infinite-dimensional groups, in the case of current groups, was studied firstly in [124], [127], [128], [137] (see [129]). An analog of the regular representation for an arbitrary infinite-dimensional group *G*, using a *G*-quasi-invariant measure on some completion $\tilde{G}$ of such a group, is defined in [141], [143].

For $X = S^1, U$ a compact or non-compact connected Lie group,Wiener measures on the loop groups $\check{G} = C(X,U)$ were constructed and their quasi-invariance were showd in [124], [127], [129], [151], [155].

Conjecture (4.1.2) was formulated by R.S. Ismagilov for the group $G = B_0^{\mathbb{N}}$ and the measure μ being the product of arbitrary one-dimensional centered Gaussian measures on the group $\tilde{G} = B^{\mathbb{N}}$ and was showd for this case in [141], [142].

The first result in this direction was showd in [156]. For the complex infinite-dimensional Borel group $Bor_0^{c,\mathbb{N}}$ and the standard Gaussian measure on its completion $Bor^{c,\mathbb{N}}$ the irreducibility of the corresponding regular representation was showd there. Here *Borc*,N 0 (respectively *Borc*,N) is the group of matrices of the form *x* = exp *t* +*s* where *t* is a diagonal matrix with a finite number of nonzero real elements (respectively arbitrary real elements) and *s* is a finite (respectively arbitrary) complex strictly upper-triangular matrix.

For the product of arbitrary one-dimensional measures on the group *B*N Conjecture(4.1.2) was showd in [144] under some technical assumptions on the measure.

In [143] Conjecture(4.1.2) was showd for the groups of the interval and circle diffeomorphisms. For the group of the interval diffeomorphisms the Shavgulidze measure [158] was used, the image of the classical Wiener measure with respect to some bijection. For the group of circle diffeomorphisms the Malliavin measure [153] was used.

Whether Conjecture(4.1.2) holds in the general case is an open problem.

In [148] it was shown that Conjecture(4.1.1) holds for the inductive limit $G = SL_0(2^\infty,\mathbb{R}) = \lim_{\to n} SL(2^{n-1},\mathbb{R})$, of the special linear groups (*simple groups*) acting on a strip of length $m \in N$ in the space of real matrices which are infinite in both directions, the measure μ being a product Gaussian measure.

Let us consider the special case of a G-space, namely the homogeneous space $X = H \setminus \check{G}$, where H is a subgroup of the group $\tilde{G}$ and μ is some quasi-invariant measure on X (if it exists) with respect to the right action R of the group G on the homogeneous space $H \setminus \tilde{G}$. In this case we call the corresponding representation $\pi^{R,\mu,H\setminus\tilde{G}}$ an analog of the quasiregular or geometric representation of the group G (see [145]).

In [125] Conjecture (4.1.1)was showd for the *solvable* infinite-dimensional real Borel group $G = Bor_0^{\mathbb{N}}$ acting on G-spaces $X^m, m \in N$, where X^m is the set of left cosets $G_m \setminus Bor^{\mathbb{N}}, and\ G_m$ is some subgroups of the group $Bor^{\mathbb{N}}$of all upper-triangular matrices of infinite order with non zero elements on the diagonal. The measure $\mu\ on\ X^m$ is the product of infinitely many onedimensional Gaussian measures on $\mathbb{R}$.

In [146], [147] Conjecture (4.1.1) was showd for the *nilpotent group* $G = B_0^{\mathbb{N}}$ and some G-spaces $X^m, m \in \mathbb{N}$, being the set of left cosets $Gm\backslash B^{\mathbb{N}}$, where G_m are some subgroups of the group $B^{\mathbb{N}}$. Here the measure $\mu\ on\ X^m$ is the infinite product of arbitrary one-dimensional Gaussian measures on R. In this case the variables $x_{pq}, 1 \le p \le q \le m$, can be approximated by linear combinations of the expressions $A_{pn}A_{qn}, q < n$, where A_{kn} are generators of one-parameter groups $\exp(tE_{kn}), k < n, t \in \mathbb{R}$.

In [126], using results of [144], we extended the results of [145]–[146] to the case of an infinite tensor product of one-dimensional non-Gaussian (general) measures.

We generalize results of [145]–[147] in another direction. Namely we show Conjecture(4.1.1) for the same nilpotent infinite-dimensional group $G = B_0^{\mathbb{N}}$ and the same G-spaces $X^m, m \in \mathbb{N}$, but with a measure μ which is the infinite tensor product of arbitrary centered Gaussian measures on $\mathbb{R}^m$, for any arbitrary fixed $m \in N$. More precisely, the measure μ on $X^m \simeq \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^{m-1} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$ is the infinite tensor product of arbitrary
Gaussian centered measures:

$$\mu = \mu_B^m = \bigotimes_{n=2}^{\infty} \mu_{B(n)},$$

where $\mu_{B(n)}$ is a Gaussian measure on the space $\mathbb{R}^{n-1}\ for\ 2 \le n \le m$ and $\mu_{B(n)}$ is a Gaussian measure on the space $\mathbb{R}^m$ for $n > m$. In this case for the approximation of the variables x_{pq} , $1 \le p \le q \le m$, we also use the commutative family of the generators $A_{kn}, 1 \le k \le m < n$, but the corresponding expressions are much more complicated. In fact the extensions of [145]–[147] to the present case are not at all simple, the above expressions are no longer polynomials in the generators A_{kn} they rather involve, next to the generators, also the one-parameter groups

$$T^{R,\mu_B^m}_{exp(tE_{kn})} = exp(tA_{kn}), t \in \mathbb{R},$$

their derivatives and very special suitable chosen combinations that allow to approximate in an appropriate way the variables involved (see Lemmas (4.1.12)and (4.1.15)
Let us consider the group $\tilde{G} = B^{\mathbb{N}}$ of all upper-triangular real matrices of infinite order with unities on the diagonal

$$\tilde{G} = B^{\mathbb{N}} = \left\{ I + x | x = \sum_{1\le k<n} x_{kn}E_{kn} \right\},$$

and its subgroup

$$G = B_0^{\mathbb{N}} = I + x \in B^{\mathbb{N}} \mid x\, is\, finite,$$

where E_{kn} is an infinite-dimensional matrix with 1 at the place $k, n \in \mathbb{N}$ and zeros elsewhere, $x = (x_{kn})_{k<n}$ is *finite* means that $x_{kn} = 0$ for all *(k, n)* except for a finite number of indices k, n.

Obviously, $B_0^{\mathbb{N}} = \lim_{\to n} B(n, \mathbb{R})$ is the inductive limit of the group $B(n, \mathbb{R})$ of real uppertriangular matrices with units on the principal diagonal

$$B(n, \mathbb{R}) = \left\{ I + \sum_{1\le k\le r\le n} x_{kr}E_{kr} | x_{kr} \in \mathbb{R} \right\}$$

with respect to the natural imbedding $B(n, \mathbb{R}) \subset B(n + 1, \mathbb{R}). For\ m \in \mathbb{N}$ we also define the subgroups G_m, respectively G^m, of the group $B^{\mathbb{N}}$ as follows:

$$G_m = \left\{ I + x \in B^{\mathbb{N}} \middle| x = \sum_{m<k<n} x_{kn}E_{kn} \right\},$$

$$G^m = \left\{ I + x \in B^{\mathbb{N}} \middle| x = \sum_{1\le k\le m, k<n} x_{kn}E_{kn} \right\}.$$

Since $B^{\mathbb{N}} = G_m \cdot G^m$ the space X^m of left cosets $X^m = G_m \backslash B^{\mathbb{N}}$ is isomorphic to the group G^m. We use the notation $G^m \simeq G_m$. By construction, the right action *R* of the group *G* is well defined on the space X^m. More precisely if we define the decomposition $x = x_m \cdot x^m$:

$$B^{\mathbb{N}} \ni x \mapsto x_m \cdot x^m \in G_m \cdot G^m,$$

the right action *R* of the group $B_0^{\mathbb{N}}$ on the space X^m is defined as follows:

$$R_t\,(x^m) = (x^m t^{-1})^m, x^m \in G^m, t \in B_0^{\mathbb{N}}.$$

Define the measure $\mu^m := \mu_B^m$ on the space $X^m \simeq Gm$

$$X^m \simeq \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times R^{m-1} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$$

by the formula $\mu_B^m = \otimes_{n=2}^{\infty} \mu_{B^{(n)}}$, where $\mu_{B^{(n)}}$ is the Gaussian measure on the space $\mathbb{R}^m$ for $n > m$ (respectively on the space $\mathbb{R}^{n-1}$ for $2 \le n \le m$) defined by

$$d\mu_{B^{(n)}}(x) = \frac{1}{\sqrt{(2\pi)^m \, detB^{(n)}}}\, exp\left(-\frac{1}{2}\left(\left(B^{(n)}\right)^{-1}x, x\right)\right)dx$$

$$= \sqrt{\frac{detC^{(n)}}{(2\pi)^m}}\, exp\left(-\frac{1}{2}\left(C^{(n)}x, x\right)\right)dx \qquad (1)$$

where $B^{(n)}$ are positive-definite operators in the space $\mathbb{R}^m\,(or\ \mathbb{R}^{n-1}), x = (x_{1n}, x_{2n}, \ldots, x_{mn}), dx$ is a Lebesgue measure on $\mathbb{R}^m\ and\ C^{(n)} = (B^{(n)})^{-1}$.

Lemma(4.1.3)[123]: For the measure μ_B^m we have

$$(\mu_B^m)^{R_t} \sim \mu_B^m, \qquad \forall t \in B_0^{\mathbb{N}}$$

(with ∼ meaning equivalence).

Proof: The right action R_t for $t \in B_0^{\mathbb{N}}$ changes linearly only a finite number of coordinates of the point $x \in X^m$.
Now we can define the representation associated with the right action

$$T^{R,\mu_B^m} : B_0^{\mathbb{N}} \to U\left(L^2\left(X^m, \mu_B^m\right)\right)$$

in the natural way, i.e.

$$\left(T_t^{R,\mu_B^m} f\right)(x) = \left(d\mu_B^m\left(R_t^{-1}(x)\right)/d\mu_B^m(x)\right)^{1/2} f\left(R_t^{-1}(x)\right).$$

Lemma(4.1.4)[123]: The measure μ_B^m on the space X^m is ergodic with respect to the right action R of the group $B_0^{\mathbb{N}}$ on the space X^m.
Proof: It is well known that any measurable function on $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdot$ with the standard Gaussian measure $\mu_I = \otimes_{n=1}^\infty \mu_{I_n}$, where $I_n \equiv I$ (see (i)) which is invariant under any change of the first coordinates (i.e. with respect to the additive action of the group R_0^∞) coincides almost everywhere with a constant function (see [159]). The proof works also in the case where we replace $\mathbb{R}$ *by* $\mathbb{R}^m, m > 1$, and the standard Gaussian measure μ_I on $\mathbb{R}$ with any probability measure $\mu_{B^{(n)}}$ *on* $\mathbb{R}^m$ equivalent with the Lebesgue measure on $\mathbb{R}^m$. To show this it is sufficient to see that any function $f \in L^1((\mathbb{R}^m)^\infty, \otimes_{n=1}^\infty \mu_{B^{(n)}})$ is the limit of $\mu k -$ *a.e.* constant functions $f^k : f = lim_k f^k$, *where* $\mu_k = \otimes_{n=1}^k \mu_{B^{(n)}}$,

$$f^k = \int_{(\mathbb{R}^m)^\infty} f(x) d\mu^k(x) \text{ and } \mu^k = \bigotimes_{n=k+1}^\infty \mu_{B^{(n)}}.$$

Therefore the proof follows from the fact that the measure $\mu_B^m = \otimes_{n=2}^\infty \mu_{B^{(n)}}$ on the space $X^m = \mathbb{R}^1 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^{m-1} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$ is the infinite tensor product of Gaussian measures $\mu_{B^{(n)}}$ on the space $\mathbb{R}^m (for\ n > m)$, from the fact that the right action $R_t\ for\ t \in B_0^{\mathbb{N}}$ changes only a finite number of coordinates of the point $x \in X^m$, and that the group $G_0^m = G^m \cap B_0^{\mathbb{N}} \subset X^m$ acts transitively on itself. In fact it is shown that the measure is ergodic with respect to the action of the subgroup $G_0^m \subset B_0^{\mathbb{N}}$.
Theorem (4.1.5)[123]: *For the measure μ_B^m the following four statements are equivalent*:
(i) *the representation T_t^{R,μ_B^m} is irreducible*;
(ii) $(\mu_B^m)^{L_t} \perp \mu_B^m \forall t \in B(m,\mathbb{R}) \setminus \{e\}$;
(iii) $(\mu_B^m)^{L_{exp(tEpq)}} \perp \mu_B^m \forall t \in \mathbb{R} \setminus \{0\} \forall 1 \le p < q \le m$;
(iv) $S_{pq}^L(\mu_B^m) = \sum_{n=q+1}^\infty c_{pp}^{(n)} b_{qq}^{(n)} = \infty \forall 1 \le p < q \le m$,
where $B^{(n)} = (b_{kr}^{(n)})_{k,r=1}^m, C^{(n)} = (c_{kr}^{(n)})_{k,r=1}^m$ *and* $C^{(n)} = (B^{(n)})^{-1}$.
Proof: The proof of Theorem (4.1.5) is organized as follows:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).$$

The parts $(i) \Rightarrow (ii) \Rightarrow (iii)$ are evident. The part $(iii) \Leftrightarrow (iv)$ follows from Lemma(4.1.9), which is based on the Kakutani criterion [138].
The idea of the proof of irreducibility, i.e. the part $(iv) \Rightarrow (i)$. Let us denote by $\mathfrak{A}^m$ the von Neumann algebra generated by the representation T_t^{R,μ_B^m}

$$\mathfrak{A}^m = \left(T_t^{R,\mu_B^m} \mid t \in G\right)''.$$

We show that $(iv) \Rightarrow [(\mathfrak{A}^m)' \subset L^\infty(X^m, u_b^m)] \Rightarrow (i)$. Let the inclusion $(\mathfrak{A}^m)'' \subset L^\infty(X^m, \mu_B^m)$ holds. Using the ergodicity of the measure μ_B^m Lemma (4.1.9) this shows the irreducibility. Indeed in this case an operator $A \in (\mathfrak{A}^m)'$ should be the operator of multiplication (since $(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)$) by some essentially bounded function $a \in L^\infty(X^m, \mu_B^m)$. The commutation relation $[A, T_t^{R,\mu_B^m}] = 0 \, \forall t \in B_0^{\mathbb{N}}$ implies $a(R_t^{-1}(x)) = a(x) \, (mod\, \mu_B^m) \, \forall t \in B_0^{\mathbb{N}}$, so by ergodicity of the measure μ_B^m with respect to the right action of the group $B_0^{\mathbb{N}}$ on the space X^m we conclude that $A = a = const \, (mod\, \mu_B^m)$. This then shows the irreducibility in Theorem(4.1.5), i.e. the part $[(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)] \Rightarrow (i)$.

The proof of the remaining part, i.e. the implication $(iv) \Rightarrow [(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)]$ is based on the fact that the operators of multiplication by independent variables $x_{pq}, 1 \le p \le m, p < q$, may be approximated in the strong resolvent sense by some functions of the generators

$$A_{kn}^{R,m} = \frac{d}{dt} \, T_{I+tE_{kn}}^{R,\mu_B^m}\Big|_{t=0}, k, n \in \mathbb{N}, k < n,$$

i.e. that the operators x_{pq} are affiliated with the von-Neumann algebra $\mathfrak{A}^m$. See Lemma (4.1.15) and Lemma (4.1.16) .

Definition (4.1.6)[123]: Recall (cf., e.g., [132]) that a non-necessarily bounded self-adjoint operator *A* in a Hilbert space *H* is said to be *affiliated* with a von Neumann algebra *M* of operators in this Hilbert space $H, if \; exp(itA) \in M$ for all $t \in \mathbb{R}$. One then writes $A\eta M$.

Since the algebra $(exp(itxpq) \mid t \in \mathbb{R}, 1 \le p \le m, p < q)''$ is the maximal abelian subalgebra in the von Neumann algebra *B(H)* of all bounded operator in the Hilbert space $H = L^2(X^m, \mu_B^m)$ we conclude that $(exp(itx_{pq}) \mid t \in \mathbb{R}, 1 \le p \le m, p < q)'' = L^\infty(X^m, \mu_B^m)$. The inclusion $(exp(itx_{pq}), 1 \le p \le m, p < q) \subset \mathfrak{A}^m \; implies \; (\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)$.

To finish the proof of Theorem(4.1.5) it remains to show the implication

$$(iv) \Rightarrow \left(x_{pq} \, \eta \, \mathfrak{A}^m, 1 \le p \le m, p < q\right) \Leftrightarrow exp(itx_{pq}) \in \mathfrak{A}^m, 1 \le p \le m, p < q$$

It is sufficient to show that $\Sigma_m > CS_m$, for some *C* >0, where

$$S_m := \sum_{1 \le p < q \le m} S_{pq}^L(\mu_B^m). \, \Sigma_m := \sum_{1 \le r \le p \le q \le m} \sum_{pq}^{r} (m),$$

and the series $S_{pq}^L(\mu^m)$ *and* $\sum_{pq}^r(m)$ are defined in Lemmas (4.1.9) and (4.1.13)). This is done in Appendices A–C.

We define the *generalization of the characteristic polynomial* for matrix *C* and establish some its properties. These properties are used then in Appendices B and C. For a matrix $C \in Mat(k, \mathbb{C})$ we set

$$G_k(\lambda) = detC_k(\lambda), where \; C_k(\lambda) = C + \sum_{r=1}^{k} \lambda_r E_{rr}, \lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k.$$

Lemma (4.1.7)[123]: Lemma (4.1.22) *For a positive definite matrix* $C \in Mat(k, \mathbb{C}), \lambda \in \mathbb{R}^k$ *with* $\lambda_r \ge 0, r = 1, \dots, k$, *we have*

$$\frac{\partial}{\partial \lambda_p} \frac{Gk(\lambda)}{G_l(\lambda)} \ge 0,$$

where $G_l(\lambda) = M_{12\dots l}^{12\dots l}(C_k(\lambda))$ *and* $1 \le p \le l \le k$.

The proof of Lemma (4.1.7) is based on the following inequality (see Lemma (4.1.21).)

Lemma (4.1.8)[123]: (Hadamard–Ficher's inequality [135], [136], see also [150]) *Let* $C \in \mathrm{Mat}(m, \mathbb{R})$ *be a positive definite matrix and* $\emptyset \subseteq \alpha, \beta \subseteq \{1, \ldots, m\}$. *Then*

$$\begin{vmatrix} detC_\alpha & detC_{\alpha\cap\beta} \\ detC_{\alpha\cup\beta} & detC_\beta \end{vmatrix} = \begin{vmatrix} M(\alpha) & M(\alpha \cap \beta) \\ M(\alpha \cup \beta) & M(\beta) \end{vmatrix} \geq 0,$$

where C_α *for* $\alpha = \{\alpha_1, \ldots, \alpha_s\}$ *denotes the matrix which entries lie on the intersection of* $\alpha_1, \ldots, \alpha_s$ *rows and* $\alpha_1, \ldots, \alpha_s$ *columns of the matrix C and* $M(\alpha) = M_\alpha^\alpha(C) = detC_\alpha$ *are corresponding minors of the matrix C.*

The "best" approximation of x_{pq} by the generators $A_{kn}^{R,m}$ is based on the exact computation of the matrix elements

$$\emptyset_p(t) = \left(T_t^{R,\mu_B^m}\mathbf{1},\mathbf{1}\right), t = I + \sum_{r=1}^{p} t_r E_{rn}, (t_r)_{r=1}^{p} \in \mathbb{R}^p,$$

of the representation T^{R,μ_B^m} and their generalization (see Appendix B, Lemma (4.1.23) , and on the finding the appropriate combinations of operator functions of the generators $A_{kn}^{R,m}$ to approximate the operators of multiplication by x_{pq}.

Finally the proof of the inequality $\Sigma_m > CS_m$, is based on Lemmas (4.1.7), (4.1.8) and(4.1.25) dealing with some inequalities involving the generalized characteristic polynomials. Lemma(4.1.25) is showd.

Lemma (4.1.9)[123]: *For the measure* μ_B^m *we have the equivalence of*

(i) $(\mu_B^m)^{L_{exp(tE_{pq})}} \perp \mu_B^m \forall t \in \mathbb{R} \setminus \{0\} \forall 1 \leq p < q \leq m$ *and*

(ii)$S_{pq}^L(\mu_B^m) = \sum_{n=q+1}^{\infty} c_{pp}^{(n)} b_{qq}^{(n)} = \sum_{n=q+1}^{\infty} \frac{c_{pp}^{(n)} A_q^q(C^{(n)})}{det(C^{(n)})} = \infty \; \forall 1 \leq p < q \leq m,$

where $B^{(n)} = (b_{kr}^{(n)})_{k,r=1}^m, C^{(n)} = (c_{kr}^{(n)})_{k,r=1}^m$ *and* $C^{(n)} = (B^{(n)})^{-1}$

Proof: The proof is based on the Kakutani criterion [138] and on the exact formula for the Hellinger integral

$$H(\mu, \nu) = \int_x \sqrt{\frac{d\mu}{d\rho}\frac{d\nu}{d\rho}}\, d\rho,$$

for two Gaussian measure $\mu = \mu B_1$ *and* $\nu = \mu B_2$ (see [149]):

$$H(\mu B_1, \mu B_2) = \left(\frac{detB_1\, detB_2}{det^2\, \frac{B_1 + B_2}{2}}\right)^{-1/4} = \left(\frac{detC_1\, detC_2}{det^2\, \frac{c_1 + c_2}{2}}\right)^{-1/4}, \quad (2)$$

where $C_i = (B_i)^{-1}, i = 1, 2.$

Let us consider the one-parameter subgroup $exp(tE_{pq}) = I + tE_{pq} \in B(m, R), 1 \leq p < q \leq m, t \in \mathbb{R}$. Using (1) we have for the positive definite operator $B = B^{(n)}$ *in* $\mathbb{R}^m$:

$$d\mu_B^{L_{I+tE_{pq}}}(x) = \sqrt{\frac{detC}{(2\pi)^m}} exp\left(-\frac{1}{2}\left(C\, exp(tE_{pq})x, exp(tE_{pq})x\right)\right) d\, exp(tE_{pq})^x$$

$$= \sqrt{\frac{detC}{(2\pi)^m}} exp\left(-\frac{1}{2}\left(exp(tE_{pq})^*, Cexp\left(tE_{pq}\right)x, x\right)\right) dx = d\mu_{B_{pq(t)}}(x)$$

where $(Bpq(t))^{-1} = C_{pq}(t) = exp(tE_{pq})^* \, C \, exp(tE_{pq})$ (we note that $det\, C = detC_{pq}(t)$). Hence, using (2) we get

$$H\mu \left(L_B^{I+tE_{pq}}, \mu B\right) = \left(\frac{detC_{pq}\,(t) detc}{det^2 \,\frac{c_{pq}(t)+c}{2}}\right)^{1/4} = \left(\frac{detC}{det^2 \,\frac{c_{pq}(t)+c}{2}}\right)^{1/2} \tag{3}$$

We shall show that

$$det\, \frac{C_{pq}(t) \; + C}{2} = \; detC \; + \frac{t^2}{4} c_{pp} A_q^q(C), \tag{4}$$

where $A_q^p(C), 1 \le p, q \le m,$ denote the cofactors of the matrix C corresponding to the row p and the column q. We have

$$\frac{det\,\frac{C_{pq}(t)+C}{2}}{detC} = \frac{detC + \frac{t^2}{4}c_{pp}A_q^q(C)}{detC} = \; 1 + \; \frac{t^2}{4} c_{pp} b_{qq},$$

hence

$$\left(\frac{detC}{det\, \frac{C_{pq}(t) \; + C}{2}}\right)^{1/2} = \left(1 + \; \frac{t^2}{4} c_{pp} b_{qq}\right)^{-1/2}$$

and finally, using (3) we get

$$H\left((\mu_B^m)^{L_{I+tE_{pq}}}, \mu_B^m\right) = \prod_{n=q+1}^{\infty} H\, \mu_{B^{(n)}}^{L_{I+tE_{pq}}}, \mu_{B^{(n)}} = \prod_{n=q+1}^{\infty} \left(1 + \; \frac{t^2}{4} c_{pp}^{(n)} b_{qq}^{(n)}\right)^{-1/2}$$

where

$$B^{(n)} \; = \sum_{1\le r,s \le m} b_{rs}^{(n)}\, E_{rs} \; and \; C^{(n)} := \left(B^{(n)}\right)^{-1} \; = \sum_{1\le r,s\le m} c_{rs}^{(n)}\, E_{rs} \; .$$

So using the properties of the Hellinger integral for two Gaussian measures we conclude that

$$(\mu_B^m)^{L_{I+tE_{pq}}} \perp \mu_B^m \; \forall t \; \in \mathbb{R} \setminus \{0\} \; \Leftrightarrow \infty \prod_{n=q+1}^{\infty} \left(1 + \; \frac{t^2}{4} c_{pp}^{(n)} b_{qq}^{(n)}\right)^{-1/2} \; = \; 0$$

$$\Leftrightarrow \; S_{pq}^L(\mu_B^m) = \infty.$$

To show (4) we set $C_{pq}(t) \; = \; exp(tE_{pq})^* C \; exp(tE_{pq})$. We have for $m \; \in \; N \; and \; 1 \; \le p \; < q \; \le \; m$ using the identity $exp(tE_{pq}) \; = \; I \; + tE_{pq}, t \; \in \; \mathbb{R}$,

$$C_{pq}(t) \; = \begin{vmatrix} c_{11} & \cdots & c_{1p} & \cdots & c_{1q} + tc_{1p} & \cdots & c_{1m} \\ c_{1p} & \cdots & c_{pp} & \cdots & c_{pq} + tc_{pp} & \cdots & c_{pm} \\ c_{1q} + tc_{1p} & \cdots & c_{pq} + tc_{pp} & \cdots & c_{qq} + 2tc_{pp} + t^2 c_{pp} & \cdots & c_{qm} + tc_{qm} \\ c_{1m} & \cdots & c_{pm} & \cdots & c_{qm} + tc_{qm} & \cdots & c_{mm} \end{vmatrix}$$

hence

$$det\,\frac{C_{pq}(t)+C}{2}$$

$$=\begin{vmatrix} c_{11} & \cdots & c_{1p} & \cdots & c_{1q}+\frac{t}{2}c_{1p} & \cdots & c_{1m} \\ c_{1p} & \cdots & c_{pp} & \cdots & c_{pq}+\frac{t}{2}c_{pp} & \cdots & c_{pm} \\ c_{1q}+tc_{1p} & \cdots & c_{pq}+\frac{t}{2}c_{pp} & \cdots & c_{qq}+2tc_{pq}+\frac{t^2}{2}c_{pp} & \cdots & c_{qm}+\frac{t}{2}c_{pm} \\ c_{1m} & \cdots & c_{pm} & \cdots & c_{qm}+\frac{t}{2}c_{pm} & \cdots & c_{mm} \end{vmatrix}$$

$$=\begin{vmatrix} c_{11} & \cdots & c_{1p} & \cdots & c_{1q} & \cdots & c_{1m} \\ c_{1p} & \cdots & c_{pp} & \cdots & c_{pq} & \cdots & c_{pm} \\ c_{1q} & \cdots & c_{pq} & \cdots & c_{qq}+\frac{t^2}{4}c_{pp} & \cdots & c_{qm} \\ c_{1m} & \cdots & c_{pm} & \cdots & c_{qm} & \cdots & c_{mm} \end{vmatrix} = \mathrm{detC}+\frac{t^2}{4}c_{pp}\mathrm{A}_{\mathrm{q}}^{\mathrm{q}}(\mathrm{C}).$$

This ends the proof of Lemma(4.1.9) , and thus also of $(iii) \Leftrightarrow (iv)$.

Approximation of the variables x_{pq}

We first show Lemmas (4.1.12) and(4.1.15) , which give a suitable approximation of x_{pq} only on the vector $f = 1 \in L^2(X^m, \mu_B^m)$

We shall also use the well-known result (see, for example, [130])

$$\min_{x\in\mathbb{R}^n}\left(\sum_{k=1}^{n} a_k x_k^2 \,\middle|\, \sum_{k=1}^{n} xk = 1\right) = \left(\sum_{k=1}^{n}\frac{1}{a_k}\right)^{-1}, a_k > 0, k = 1,2,\ldots,n.$$

We use the same result in a slightly different form with $b_k \neq 0, k = 1,2,\ldots,n,$

$$\min_{x\in\mathbb{R}^n}\left(\sum_{k=1}^{n} a_k x_k^2 \,\middle|\, \sum_{k=1}^{n} x_k b_k = 1\right) = \left(\sum_{k=1}^{n}\frac{b_k^2}{a_k}\right)^{-1} \tag{5}$$

The minimum is realized for

$$xk = \frac{b_k}{a_k}\left(\sum_{k=1}^{n}\frac{b_k^2}{a_k}\right)^{-1}.$$

For any subset $I \subset \mathbb{N}$ let us denote as before by $\langle f_n \mid n \in I\rangle$ the closure of the linear space generated by the set of vectors $(f_n \mid n \in I)$ in a Hilbert space H.

We note that the distance $d(f_{n+1}; \langle f_1,\ldots,f_n\rangle)$ of the vector f_{n+1}in H from the hyperplane $\langle f_1,\ldots,f_n\rangle$ may be calculated in terms of the Gram determinants $\Gamma\,(f_1,f_2,\ldots,f_k)$ corresponding to the set of vectors $f_1,f_2,\ldots,f_k$ (see [133]):

$$d\, f_{n+1}; \langle f_1,\ldots,f_n\rangle = \min_{t=(t_k)\in\mathbb{R}^n}\left\|f_{n+1}+\sum_{k=1}^{n} t_k f_k\right\|^2 = \frac{\Gamma(f_1,f_2,\ldots,f_{n+1})}{\Gamma\,(f_1,f_2,\ldots,f_n)}, \tag{6}$$

where the Gram determinant is defined by $\Gamma\,(f_1,f_2,\ldots,f_n) = det\,\gamma\,(f_1,f_2,\ldots,f_n)\ and\ \gamma\,(f_1,f_2,\ldots,f_n) =: \gamma_n$ is the Gram matrix

$$\gamma\,(f_1,f_2,\ldots,f_n) = \begin{pmatrix} (f_1,f_1) & (f_1,f_2) & \cdots & (f_1,f_n) \\ (f_2,f_1) & (f_2,f_2) & \cdots & (f_2,f_n) \\ & & \cdots & \\ (f_n,f_1) & (f_n,f_2) & \cdots & (f_n,f_n) \end{pmatrix}.$$

Lemma (4.1.10)[123]: We have $d(f_{n+1};\langle f_1,\dots,f_n\rangle)=\frac{\det\gamma_{n+1}}{\det\gamma_n}=(f_{n+1},f_{n+1})-(\gamma_n^{-1}\ d_{n+1},d_{n+1})$, where $d_{n+1}=((f_1,f_{n+1}),(f_2,f_{n+1}),\dots,(f_n,f_{n+1}))\in\mathbb{R}^n$.

Proof: We may write

$$\left\|\sum_{k=1}^{n}t_kf_k-f_{n+1}\right\|^2=\sum_{k,m=1}^{n}t_kt_m(f_k,f_k)-2\sum_{k=1}^{n}t_k(f_k,f_{n+1})+(f_{n+1},f_{n+1})$$
$$=(\gamma_nt,t)-2(t,d_{n+1})+(f_{n+1},f_{n+1}),$$

where $t=(t_1,t_2,\dots,t_n)\in\mathbb{R}^n$. Using (58) for $A_n=\gamma_n$ we get

$$(\gamma_nt,t)-2(t,d_{n+1})=(\gamma_n(t-t_0),(t-t_0))-(\gamma_n^{-1}\ d_{n+1},d_{n+1}),$$

where $t_0=\gamma_n^{-1}\ d_n$. Hence we get (see (6))

$$\min_{t=(t_k)\in\mathbb{R}^n}\left\|f_{n+1}-\sum_{k=1}^{n}t_kf_k\right\|^2\min_{t=(t_k)\in\mathbb{R}^n}(\gamma_nt,t)-2(t,d_{n+1})+(f_{n+1},f_{n+1})$$
$$=(f_{n+1},f_{n+1})(\gamma_n^{-1}\ d_{n+1},d_{n+1})+\min_{t=(t_k)\in\mathbb{R}^n}(\gamma_n(t-t_0),(t-t_0))$$
$$=(f_{n+1},f_{n+1})(\gamma_n^{-1}\ d_{n+1},d_{n+1}).$$

Remark (4.1.11)[123]: In fact a more general result holds. Let us denote by A_{n+1} the real non-necessarily symmetric matrix in R^{n+1} and by A_n its $n\times n$ block after crossing the element in the last column and row, by $v_{n+1}=(a_{1n+1},a_{2n+1},\dots,a_{nn+1}),h_{n+1}=(a_{n+11},a_{n+12},\dots,a_{n+1n})$ vectors $v_{n+1},h_{n+1}\in\mathbb{R}^n$. If $detA_n\neq 0$ then we have

$$a_{n+1n+1}-(A_n^{-1}\ v_{n+1},h_{n+1})=\frac{detA_{n+1}}{detA_n}. \tag{7}$$

Proof: It is sufficient to use the identity (Schur–Frobenius decomposition)

$$A_{n+1}=\begin{pmatrix}A_n & v_{n+1}^t\\ h_{n+1} & a_{n+1n+1}\end{pmatrix}=\begin{pmatrix}A_n & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}Id & A_n^{-1}\ v_{n+1}^t\\ h_{n+1} & a_{n+1n+1}\end{pmatrix}.$$

The generators

$$A_{kn}:=A_{kn}^{R,m}=\frac{d}{dt}T_{I+tE_{kn}}^{R,\mu_B^m}\Big|_{t=0}$$

of the one-parameter groups $I+tE_{kn}$ have the following form (on smooth functions of compact support):

$$A_{kn}=\sum_{r=1}^{k-1}x_{rk}D_{rn}+D_{kn},1\le k\le m,k<n,\qquad A_{kn}=\sum_{r=1}^{m}x_{rk}D_{rn},m<k<n,$$

where

$$D_{kn}=\frac{\partial}{\partial x_{kn}}-\frac{1}{2}\left(x,\left(B^{(n)}\right)^{-1}E_{kn}\right),1\le k<n. \tag{8}$$

To simplify the further computations let us consider the corresponding Fourier transforms F_m in the variables $x_{kn},1\le k\le m,m<n$,

$$F_m:L^2(X^m,\mu_B^m)\to L^2(X^m,\mu_C^m).$$

We have

$$F_mD_{kn}F_m^{-1}=iy_{kn}\ for\ (k,n),1\le k\le m,m<n,and\ F_m1=1.$$

Let us set $\mu_C=\otimes_{n=2}^{\infty}\mu_{C^{(n)}}$ with $C^{(n)}=B^{(n)}$ for $2\le n\le m$ and $C^{(n)}=(B^{(n)})^{-1}\ for\ n>m$.

We define the Fourier transform F_m as the infinite tensor product $F_m = \otimes_{n=m+1}^{\infty} F_{mn}$ where

$$F_{mn} : L^2\left(\mathbb{R}^m, \mu_{B^{(n)}}\right) \to L^2\left(\mathbb{R}^m, \mu_{C^{(n)}}\right)$$

is the image of the standard Fourier transform F_m in the space $L^2(\mathbb{R}^m, dx), i.e. F_{mn} = U(C^{(n)})^{-1} F^m U(B^{(n)})$, where

$$U\left(B^{(n)}\right) = \left(\frac{d\mu_{B^{(n)}}(x)}{dx}\right)^{\frac{1}{2}} \begin{array}{ccc} L^2\left(\mathbb{R}^m, \mu_{B^{(n)}}\right) & \xrightarrow{F_{mn}} & L^2\left(\mathbb{R}^m, \mu_{C^{(n)}}\right) \\ \downarrow & & \downarrow \\ L^2(\mathbb{R}^m, dx) & \xrightarrow{F^m} & L^2(\mathbb{R}^m, dx) \end{array} U\left(C^{(n)}\right)\left(\frac{d\mu_{C^{(n)}}(x)}{dx}\right)^{1/2}$$

Since the standard Fourier transform F_m is defined as follows:

$$(F^m f)(y) = \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} exp\, i(y,x) f\,(x) dx\,,$$

and, for $D = B^{(n)}$ respectively $D = C^{(n)}$

$$U(D) = \left(\frac{d\mu D(x)}{dx}\right)^{1/2} = \frac{1}{((2\pi)^m\, detD^{1/4})}\, exp\left(-\frac{1}{4}(D^{-1}x, x)\right),$$

we have finally for F_{mn}:

$$(F_{mn} f\,)(y) = \left(U\left(C^{(n)}\right)^{-1} F^m U\left(B^{(n)}\right) f\right)(y)$$

$$= \frac{1}{((2\pi)^m\, detC^{(n)})^{1/4}}\, exp\left(\frac{1}{4}\left(\left(C^{(n)}\right)^{-1} y, y\right)\right) \frac{1}{\sqrt{(2\pi)^m}}$$

$$\int_{\mathbb{R}^m} exp\, i(y,x) f\,(x) dx\ \left((2\pi)^m\, detB^{(n)}\right)^{1/4} exp\left(-\frac{1}{4}\, {B^{(n)}}^{-1}\, x, x\right) dx$$

$$= \frac{exp(\frac{1}{4}\ ((C^{(n)})^{-1} y, y))}{\sqrt{(2\pi)^m\, detC^{(n)}}} \int_{\mathbb{R}^m} exp\ \left(i(y,x) - \frac{1}{4}\left(\left(B^{(n)}\right)^{-1} x, x\right)\right) f\,(x) dx\,.$$

Using Fourier transform F_{m} we obtain for $\widetilde{A_{kn}} = F_m A_{kn} (F_m)^{-1}$:

$$\widetilde{A_{kn}} = i\left(\sum_{r=1}^{k-1} x_r k y_{rn} + y_{kn}\right), 1 \le k \le m < n,$$

$$\widetilde{A_{kn}} = \sum_{r=1}^{m} D_{rk}(y) y_{rn}\ , m < k < n, \tag{9}$$

where

$$D_{kn}(y) = \frac{\partial}{\partial y_{kn}} - \frac{1}{2}\left(x, \left(C^{(n)}\right)^{-1} E_{kn}\right), 1 \le \_k < n.$$

Let us set for $s = (s_1, \ldots, s_r) \in Rr\ and\ 1 \le r \le p < q \le m$

$$\xi_n^{rp}(s) = F_m\left(D_{pn}\, exp\left(\sum_{l=1}^{r} s_l A_{ln}\right)\right) 1 = i y_{pn}\, exp\left(\sum_{l=1}^{r} s_l \widetilde{A_{ln}}\right) 1. \tag{10}$$

For a function $f : X^m \to \mathbb{C}$ we set

$$Mf = \int_{X^m} f(x) d\mu_B^m(x).$$

To approximate the variables $x_{pq}, 1 \leq p < q \leq m$, we use

Lemma(4.1.12)[123]: Let $1 \leq r \leq p < q \leq m$. For any $s^{(n)} = (s_1^{(n)}, \dots, s_r^{(n)}) \in \mathbb{R}^r$, and for any $\alpha^{(n)} = (\alpha_1^{(n)}, \dots, \alpha_m^{(n)}) \in \mathbb{R}^m, n \in \mathbb{N}$, we have

$$x_{pq} \in \langle exp\left(\sum_{l=1}^{r} s_l^{(n)} A_{ln}\right)\left(\sum_{k=1}^{m} \alpha_k^{(n)} A_{kn}\right) 1 | n \in \mathbb{N}, m < n\rangle \Leftrightarrow \Sigma_{pq}^r(s, \alpha, m) = \infty,$$

where $s = (s^{(n)})_{n=m+1}^{\infty}, \alpha = (\alpha^{(n)})_{n=m+1}^{\infty}, \alpha_q^{(n)} = 1$ and

$$\Sigma_{pq}^r(s, \alpha, m) = \sum_{n=m+1}^{\infty} \frac{|M\xi_n^{rp}(s^{(n)})|^2}{c_{pp}^{(n)} - |M\xi_n^{rp}(s^{(n)})|^2 + \left\|(A_{qn} - x_{pq}D_{pn} + \sum_{k=1, k\neq p}^{m} \alpha_k^{(n)} A_{kn})1\right\|^2}. \quad (11)$$

Before proving Lemma(4.1.12) let us make some comments about the procedure for arriving at the expressions used for the approximation of the variables x_{pq} on the left-hand side of the equivale.

Proof: If we put $\sum_n t_n M\xi_n^{rp}(s^{(n)}) = 1$ we get

$$\left\|\left[\sum_n t_n \, exp\left(\sum_{l=1}^{r} s_l^{(n)} A_{ln}\right)\left(\sum_{k=1}^{m} \alpha_k^{(n)} A_{kn}\right) - x_{pq}\right] 1\right\|^2$$

$$= \left\|\left[\sum_n t_n \, exp\left(\sum_{l=1}^{r} s_l^{(n)} A_{ln}\right)\left(A_{qn} - x_{pq}D_{pn} + x_{pq}D_{pn} + \sum_{k=1, k\neq q}^{m} \alpha_k^{(n)} A_{kn}\right) - x_{pq}\right] 1\right\|^2$$

$$= \left\|\sum_n t_n \left[x_{pq}\left(D_{pn} exp\left(\sum_{l=1}^{r} s_l^{(n)} A_{ln}\right) - M\xi_n^{rp}(s^{(n)})\right)\right.\right.$$

$$\left.\left. + exp\left(\sum_{l=1}^{r} s_l^{(n)} A_{ln}\right)\left(A_{qn} - x_{pq}D_{pn} + \sum_{k=1, k\neq q}^{m} \alpha_k^{(n)} A_{kn}\right)\right] 1\right\|^2$$

$$= \sum_n t_n^2 \left[\left\|x_{pq}\right\|^2 \left\|\left(D_{pn} exp\left(\sum_{l=1}^{r} s_l^{(n)} A_{ln}\right) - M\xi_n^{rp}(s^{(n)})\right)\right.\right.$$

$$\left.\left. + exp\left(\sum_{l=1}^{r} s_l^{(n)} A_{ln}\right)\left(A_{qn} - x_{pq}D_{pn} + \sum_{k=1, k\neq q}^{m} \alpha_k^{(n)} A_{kn}\right) 1\right\|^2\right]$$

$$= \sum_n t_n^2 \left[\left\|x_{pq}\right\|^2 \left(c_{pp}^{(n)} - \left|M\xi_n^{rp}\,(s^{(n)})\right|^2\right) \right.$$
$$\left. + \left\| +exp\left(\sum_{l=1}^{r} s_l^{(n)}\, A_{ln}\right)\left(A_{qn} - x_{pq}D_{pn} + \sum_{k=1,k\neq q}^{m} \alpha_k^{(n)}\, A_{kn}\right)1\right\|^2\right]$$

where we have used the equality $\|\xi - M\xi\|^2 = \|\xi\|^2 - |M\xi|^2$:

$$\# \left\|\left[D_{pn}\, exp\left(\sum_{l=1}^{r} s_l^{(n)}\, A_{ln}\right) - M\xi_n^{rp} s^{(n)}\right] 1\right\|^2 = \left\|D_{pn}1\right\|^2 - \left|M\xi_n^{rp}\left(s^{(n)}\right)\right|^2$$
$$= c_{pp}^{(n)} - \left|M\xi_n^{rp}\left(s^{(n)}\right)\right|^2.$$

Remark(4.1.13)[123]: The operator $A_{qn} = \sum_{r=1}^{q-1} x_{rq}D_{rn} + D_{qn}$ contains x_{pq} for r = p.b. Since $MD_{pn}1 = 0$ *and* $MD_{pn}\, exp(sA_{pn})1 \neq 0$ we may first think of considering $exp(sA_{pn})A_{qn}1, 1 \leq p < q \leq m$ (similarly as in [146], [147] where the linear combinations of $A_{pn}A_{qn}$ were used). But this is not sufficient for the approximation. We might then try to consider the expression

$$exp(sA_{pn})\left(\sum_{k=1}^{m} \alpha_k A_{kn}\right) \;,1 \leq p < m < n,$$

with $\alpha_q = 1$. The calculations show again that these combinations are still not sufficient to approximate x_{pq} . We arrive then at the suggestion to take

$$exp\left(\sum_{l=1}^{r} s_l A_{ln}\right)\;\left(\sum_{k=1}^{m} \alpha_k A_{kn}\right) \;,1 \leq r \leq p < q \leq m < n,$$

which is the choice made in Lemma (4.1.12).

c. For approximation of the variable x_{pq} we use p different combinations, corresponding to $\Sigma_{pq}^{r}\,(s,\alpha,m), 1 \leq r \leq p.$ All these combinations are essential, i.e. none of them can be omitted. This can be seen by constructing corresponding counterexamples and is in a contrast to the previous cases considered in [146], [147] where only one combination of $A_{pn}A_{qn}$ were used to approximate xpq.

d. To make the expression $\Sigma_{pq}^{r}\,(s,\alpha,m)$ in (11) larger (to apply then the criterium in Lemma (4.1.12) we chose $s^{(n)} \in \mathbb{R}^r$ such that

$$\left|M\xi_n^{rp}\left(s^{(n)}\right)\right|^2 = \max_{s\in\mathbb{R}^r}\left|M\xi_n^{rp}(s)\right|^2$$

(which is possible, $\left|M\xi_n^{rp}(s)\right|^2$being continuous and bounded).

e. With the same aim we chose $\alpha_k^{(n)}$ in such a way that

$$\left\| \left(A_{qn} - x_{pq} D_{pn} + \sum_{k=1, k\neq q}^{m} \alpha_k^{(n)} A_{kn} \right) 1 \right\|^2$$

$$= \min_{(t_k)\in\mathbb{R}^{m-1}} A_{qn} - x_{pq} D_{pn} + \left\| \sum_{k=1, k\neq q}^{m} t_k A_{kn} \right\|^2 .$$

f. The right-hand side of the previous expression is equal (see (6)) to

$$\frac{\Gamma\,(\mathrm{g}_1, \mathrm{g}_2, \ldots, \mathrm{g}_q^p, \ldots, \mathrm{g}_m)}{\Gamma\,(\mathrm{g}_1, \mathrm{g}_2, \ldots, \mathrm{g}_{q-1}, \mathrm{g}_{q-1}, \ldots, \mathrm{g}_m)},$$

Where

$$gk := g_{kn} := A_{kn}1, 1 \leq k \leq m, k \neq q, \mathrm{g}_q^p := \mathrm{g}_{qn}^p := (A_{qn} - x_{pq} D_{pn})1. \tag{12}$$

Definition (4.1.14)[123]: We shall say that two series $\sum_n a_n\ and \sum_n b_n$ bn with positive members are equivalent and shall denote this by $\sum_n a_n \sim \sum_n b_n$ if they are convergent or divergent simultaneously. We note that if $a_n > 0, b_n > 0, n \in \mathbb{N}$, then we have

$$\sum_{n\in\mathbb{N}} \frac{a_n}{a_n + b_n} \sim \sum_{n\in\mathbb{N}} \frac{a_n}{b_n}. \tag{13}$$

Using (5) we get, setting $b = (M\xi_n^{rp}\,(s^{(n)}))_{n=m+1}^{m+1+N} \in R^N, N \in \mathbb{N}$,

$$\min_{t\in\mathbb{N}^N} \left(\left\| \left[\sum_{n=m+1}^{m+1+N} t_n \exp\left(\sum_{l=1}^{r} s_l^{(n)} A_{ln} \right) \left(\sum_{k=1}^{m} \alpha_k^{(n)} A_{kn} \right) - x_{pq} \right] 1 \right\|^2 |(t,b) = -1 \right)$$

$$\sim \left(\sum_{n=m+1}^{m+1+N} \frac{\left| M\xi_n^{rp}(s^{(n)}) \right|^2}{c_{pp}^{(n)} - \left| M\xi_n^{rp}(s^{(n)}) \right|^2 + \left\| \left(A_{qn} - x_{pq} D_{pn} + \sum_{k=1,k\neq p}^{m} \alpha_k^{(n)} A_{kn} \right) 1 \right\|^2} \right)^{-1}$$

Due to we shall write C (respectively $\hat{C}$) instead of $C^{(\mathrm{n})}$ (respectively $\hat{C}^{(\mathrm{n})}$), where

$$C^{(n)} = \begin{pmatrix} c_{11}^{(n)} & c_{12}^{(n)} & \cdots & c_{1m}^{(n)} \\ c_{12}^{(n)} & c_{22}^{(n)} & \cdots & c_{2m}^{(n)} \\ & & \ddots & \\ c_{1m}^{(n)} & c_{2m}^{(n)} & \cdots & c_{mm}^{(n)} \end{pmatrix}$$

$$\hat{C}^{(n)} = \begin{pmatrix} c_{11}^{(n)} & c_{12}^{(n)} & \cdots & c_{1m}^{(n)} \\ c_{12}^{(n)} & c_{11}^{(n)} + c_{22}^{(n)} & \cdots & c_{2m}^{(n)} \\ & & \ddots & \\ c_{1m}^{(n)} & c_{2m}^{(n)} & \cdots & c_{11}^{(n)} + c_{22}^{(n)} + \cdots + c_{mm}^{(n)} \end{pmatrix}$$

Using this remark, notation (13) and Fourier transforms we conclude that

$$\Gamma\,(\mathrm{g}_1, \mathrm{g}_2, \ldots, \mathrm{g}_m) = \det \hat{C}, i.e. \Gamma\,(\mathrm{g}_{1n}, \mathrm{g}_{2n}, \ldots, \mathrm{g}_{mn}) = \det \hat{C}^{(n)}, \tag{14}$$

since $(\mathrm{g}_q, \mathrm{g}_p) = (\hat{C})_{pq}, 1 \leq p, q \leq m$. Indeed for $p \neq q$ we have $(\mathrm{g}_{qn}, \mathrm{g}_{pn}) = (\mathrm{g}_{pn}, \mathrm{g}_{qn}) = \left(\sum_{r=1}^{p-1} x_{rp} y_{rn} + y_{pn}, \sum_{s=1}^{q-1} x_{sq} y_{sn} + y_{qn} \right) = (y_{pn}, y_{qn}) = c_{pq}^{(n)}$,

$$(g_{qn}, g_{pn}) = \left\| \sum_{r=1}^{p-1} x_{rp} y_{rn} + y_{pn} \right\|^2 = \sum_{r=1}^{p-1} \|x_{rp}\|^2 \|y_{rn}\|^2 + \|y_{pn}\|^2 = \sum_{r-1}^{p} c_{rr}^{(n)} = \left(\hat{C}^{(n)}\right)_{pp}$$

(we reinserted here the upper index n in $c_{pq}^{(n)}$ for clarity).

In the following we shall need a variant of Lemma (4.1.9)replacing the $|M\xi_n^{rp}(s)|$ by its maximum Ξ_n^{rp} . Let us set (see (10) for definition of $\xi_n^{rp}(s)$)

$$\Xi_n^{rp} = \max_{s\in\mathbb{R}^r} |M\xi_n^{rp}(s)|^2. \tag{15}$$

Now we see that using s and α as in parts 4 and 5 of we have

$\Sigma_{pq}^r(s, \alpha, m)$

$$= \sum_n \frac{\max_{s^{(n)}\in\mathbb{R}^r} |M\xi_n^{rp}(s^{(n)})|^2}{c_{pp}^{(n)} - \max_{s^{(n)}\in\mathbb{R}^r} |M\xi_n^{rp}(s^{(n)})|^2 + \left\|(A_{qn} - x_{pq}D_{pn} + \sum_{k=1,k\neq p}^m \alpha_k^{(n)} A_{kn})1\right\|^2}$$

$$\overset{(13)}{\sim} \sum_n \frac{\max_{s^{(n)}\in\mathbb{R}^r} |M\xi_n^{rp}(s^{(n)})|^2}{c_{pp}^{(n)} + \left\|(A_{qn} - x_{pq}D_{pn} + \sum_{k=1,k\neq p}^m \alpha_k^{(n)} A_{kn})1\right\|^2}$$

$$\overset{(15)}{} \sum_n \frac{\Xi_n^{rp}}{c_{pp}^{(n)} + \Gamma\left(g_{1n}, g_{2n}, \ldots, g_{q-1n}, g_{q+1n} \cdots g_{mn}\right)}$$

$$\overset{\text{Remark (4.1.13)}}{=} \sum_n \frac{\Xi^{rp}\Gamma\left(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \cdots g_m\right)}{c_{pp}\Gamma\left(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \cdots g_m\right) + \Gamma\left(g_1, g_2, \ldots, g_q^p, \ldots, g_m\right)}$$

$$= \Sigma_{pq}^r(m) := \sum_n \frac{\Xi^{rp}\Gamma\left(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \cdots g_m\right)}{c_{pp}\Gamma(g_1, g_2, \ldots, g_m)} \overset{(14)}{=} \sum_n \frac{\Xi^{rp} A_q^q \hat{C}^{(n)}}{\det\hat{C}^{(n)}}$$

For the latter equality we have used the fact that

$$c_{pp}\Gamma\,(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m) + \Gamma\, g_1, g_2, \ldots, g_q^p, \ldots, g_m$$
$$= \Gamma\,(g_1, g_2, \ldots, g_m),$$

which follows from (25). Indeed it is sufficient to take in (25) $C = \hat{C} - c_{pp}E_{qq}$ and $\lambda_q = c_{pp}$. Then we have

$$\Gamma\,(g_1, g_2, \ldots, g_m) = \det\hat{C} = \det(\hat{C} - c_{pp}E_{qq} + c_{pp}E_{qq})$$
$$= \det(\hat{C} - c_{pp}E_{qq}) + c_{pp}A_q^q\,(\hat{C} - c_{pp}E_{qq})$$
$$= \Gamma\,(g_1, g_2, \ldots, g_q^p, \ldots, g_m) + c_{pp}\Gamma\,(g_1, g_2, \ldots, g_{q-1}, g_{q+1}, \ldots, g_m).$$

So we have show d the following lemma.

Lemma(4.1.15)[123]: Let $1 \le r \le p < q \le m$. Then for some $s_l = (s_l^{(n)})_{n=m+1}^\infty$, $\alpha_k = (\alpha_k^{(n)})_{n=m+1}^\infty$, wheres$(n)_l^{(n)}$, $\alpha_k^{(n)} \in \mathbb{R}$, $1 \le l \le r$, $1 \le k \le m$, we have

$$x_{pq} \in \langle \exp\left(\sum_{l=1}^r s_l^{(n)} A_{\ln)}\right)\left(\sum_{k=1}^m \alpha_k^{(n)} A_{kn}\right) 1 | n \in \mathbb{N}, m < n\rangle$$

$$\Leftrightarrow \ \Sigma_{pq}^{r}(m) = \sum_{n} \frac{\Xi^{rp} A_q^q \hat{C}^{(n)}}{det\hat{C}^{(n)}} = \infty. \qquad (16)$$

The proof of $(iv) \Rightarrow (x_{pq}\ \mathfrak{A}^m\ , 1 \le\ p \le\ m, p\ <\ q)$ in Theorem (4.1.5)

Idea.We show firstly that $x_{pq}\ \mathfrak{A}^m$ Am for some (p, q): $1\ \le p\ < q\ \le m$ if conditions (iv) are valid. Further we show that this also holds for all such (p, q). For this it is sufficient to show that

$$\Sigma_m\ > CS_m\, for\ some\ \ C\ > 0, \qquad (17)$$

where

$$S_m := \sum_{1 \le p < q \le m} S_{pq}^{L}(\mu^m)\, , and\ \Sigma_m := \sum_{1 \le r \le p < q \le m} \Sigma_{pq}^{r}(m) \qquad (18)$$

(see (15) for the definition of $\Sigma_{pq}^{r}(m)$). Indeed, in this case $S_m = \infty$ since $S_{pq}^{L}(\mu^m) = \infty \forall p, q\colon 1\ \le p\ < q \le\ m$ by Lemma(4.1.9)hence $\Sigma_m = \infty$by (17) and finally we conclude that $\Sigma_{pq}^{r}(m) = \infty$ for some $r, p, q\colon 1\ \le r \le p\ < q\ \le\ m$. By Lemma (4.1.15) we get that $x_{pq}\ \eta\ \mathfrak{A}^m$.

We define the generalization of the characteristic polynomial for matrix $C\ \in\ Mat(m, \mathbb{C})$ and establish some its properties. These properties are used.

We estimate $\Xi_n^{pq}\ =\ \max_{t \in \mathbb{R}^p} |M\xi_n^{rp}\ (s^{(n)})|^2$. This estimation is based on Lemma (4.1.23) which gives us the exact formula for

$$M\xi_n^{pq}(t)\ = \left(D_{qn} T^{R,\mu_B^m}_{exp(\sum_{r=1}^{p} t_r E_{rn})} 1,1\right), t =\ (t_1, t_2, \ldots, t_p)\ \in \mathbb{R}^p, 1\ \le\ p \le\ m$$

(see (43)), where D_{kn} is defined in (8). The latter formula is based of the exact formulas for the matrix elements

$$\emptyset_p(t) := \emptyset_p^{(n)}(t)\ = \left(T^{R,\mu_B^m}_{exp(\sum_{r=1}^{p} t_r E_{rn})} 1,1\right), t =\ (t_r)_{r=1}^{p}\ \in \mathbb{R}^p, 1\ \le\ p \le\ m$$

(see (37)) and theirs generalizations (see (41)). We cannot calculate explicitly

$$\Xi_n^{pq}\ =\ \max_{t \in \mathbb{R}^p} |M\xi_n^{pq}\ (t)|^2$$

but we are able by Lemmas (4.1.23) and (4.1.24) to obtain the estimation $\Xi_n^{pq}\ > \Psi_n^{pq}$,

$$\Psi_n^{pq}\ := \frac{(M_{12\ldots p-1q}^{12\ldots p-1p}\ (C_{p,q}^{(n)}\ ))2\ exp(-1)}{\left(M_{12\ldots p-1}^{12\ldots p-1}\ \left(C_p^{(n)}\ \right)\right)\left(M_{12\ldots p}^{12\ldots p}\ \left(C_p^{(n)}\ \right)\right) + \sum_{k=2}^{p} \lambda_k (A_k^p\ (C_p^{(n)}\ ))^2}$$

(see (45) and (46)). The crucial for proving (17) is Lemma (4.1.25) dealing with some inequalities involving the generalized characteristic polynomials. We use the notations of Lemma(4.1.9) :

$$S_{pq}^{L}(\mu_B^m) = \sum_{n=q+1}^{\infty} c_{pp}^{(n)}\ b_{qq}^{(n)} = \sum_{n=q+1}^{\infty} \frac{c_{pp}^{(n)}\ A_{qq}^{(n)}\left(C_m^{(n)}\right)}{detC_m^{(n)}} = \sum_{n=q+1}^{\infty} \frac{c_{pp}\ A_{qq}^{(n)}(c_m)}{detc_m}$$

Let

$$\lambda = (\lambda_k)_{k=1}^{m} \in \mathbb{R}^m, \hat{\lambda}\ = (\hat{\lambda}_k)_{k=1}^{m} = 1, \hat{\lambda}_1 = 0, \hat{\lambda}_k = \sum_{r=1}^{k-1} c_{rr}\ , 2 \le\ k \le\ m, \qquad (19)$$

$$f_q = e \sum_{1\le r\le p<q} \Psi^{rp}, 2 \le q \le m, \ f_2 = e\Psi^{11} = c_{11},$$

$$f_3 = e(\Psi^{11} + \Psi^{12} + \Psi^{22}), \dots, \tag{20}$$

$$C_m = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{12} & c_{11}+c_{22} & \cdots & c_{2m} \\ & & \ddots & \\ c_{1m} & c_{2m} & \cdots & c_{11}+\cdots+c_{mm} \end{pmatrix}$$

$$C_m = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{12} & c_{22} & \cdots & c_{2m} \\ & & \ddots & \\ c_{1m} & c_{2m} & \cdots & c_{mm} \end{pmatrix} \tag{21}$$

Obviously, we have $\hat{C}_m = C_m(\hat{\lambda}), where\ \hat{\lambda} \in \mathbb{C}^m$, is defined in (19)and we use the notation $C_m(\lambda) := C_m + \sum_{k=1}^m \lambda_k E_{kk}$.

We have the following expressions for $S_m\ and\ \Sigma_m$:

$$S_m := \sum_{1\le r<k\le m} S_{rk}^L(\mu^m) \sim \sum_{n=m+1}^{\infty} \frac{\sum_{k=2}^m \left(\sum_{r=2}^{k-1} c_{rr}\right) A_k^k(C_m)}{\det C_m} = \sum_{n=m+1}^{\infty} \frac{\sum_{k=2}^m \hat{\lambda}_k A_k^k(C_m)}{\det C_m}$$

We have replaced the series

$$S_{pq}^L(\mu_B^m) = \sum_{n=q+1}^{\infty} c_{pp}^{(n)}\, b_{qq}^{(n)}$$

with the equivalent one

$$S_{pq}^L(\mu_B^m) \sim \sum_{n=m+1}^{\infty} c_{pp}^{(n)}\, b_{qq}^{(n)}$$

If we use the equality $\hat{C}_m = C_m(\hat{\lambda})$, we get

$$\Sigma_m := \sum_{1\le r\le p<q\le m} \Sigma_{pq}^r(m) = \sum_{2\le q\le m} \sum_{1\le r\le p<q} \Sigma_{pq}^r(m) = \sum_{2\le q\le m} \sum_{1\le r\le p<q} \sum_{n} \frac{\Xi_n^{pq} A_q^q\left(\hat{C}_m^{(n)}\right)}{det\hat{C}_m^{(n)}}$$

$$= \sum_n \frac{\sum_{q=2}^m \left(\sum_{1\le r\le p<q} \Xi rp\right) A_q^q\left(C_m(\hat{\lambda})\right)}{detC_m(\hat{\lambda})} \overset{(47)}{>} \sum_n \frac{\sum_{q=2}^m \left(\sum_{1\le r\le p<q} \Psi^{rp}\right) A_q^q\left(C_m(\hat{\lambda})\right)}{detC_m(\lambda)}$$

$$\overset{(20)}{=} \sum_n \frac{e^{-1}\sum_{q=2}^m f_q A_q^q(C_m(\hat{\lambda}))}{detC_m + \sum_{q=2}^m \hat{\lambda}_q A_q^q(C_m(\hat{\lambda}|q|))} \tag{22}$$

The implications $S_m = \infty \Rightarrow \Sigma_m = \infty$ is based on the equality (see (22))

$$A_k^k\left(C_m\left(\lambda^{[k]}\right)\right) = A_k^k(C_m) + \sum_{r=1<k<i1<i_2<\cdots<i_r<m}^{m-k} \sum \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_r} A_{k\,i_1\,i_2\dots i_r}^{k\,i_1\,i_2\dots i_r}(C_m) \tag{23}$$

and on the following lemma.

Lemma (4.1.16)[123]: If $S_{kn}^{L}\,(\mu_B^m) = \infty$ for some $1 \leq k < n \leq m$ then one of the series $\Sigma_{pq}^{r}\,(m), 1 \leq r \leq p < q \leq m$, is divergent and hence by Lemma(4.1.25) we can approximate the corresponding variable x_{pq} .

Further we can approximate the remaining variables $x_{kn}, 1 \leq k \leq m < n$, as in [146]. This implies the inclusion $(\mathfrak{A}^m)' \subset L^\infty(X^m, \mu_B^m)$ and so the irreducibility of the representation (see "The idea of the proof of irreducibility" at the beginning).

We define $G_m(\lambda)$ the generalization of the characteristic polynomial $p_C(t) = det(tI - C), t \in \mathbb{C}$, of the matrix $C \in Mat(m, \mathbb{C})$:

$$G_m(\lambda) = detC_m(\lambda), \lambda \in \mathbb{C}^m, where\ C_m(\lambda) = C + \sum_{k=1}^{m} \lambda_k E_{kk}\,. \tag{24}$$

We denote by $M_{k\,i_1\,i_2 \dots i_r}^{k\,i_1\,i_2 \dots i_r}(C)$ (respectively $A_{k\,i_1\,i_2 \dots i_r}^{k\,i_1\,i_2 \dots i_r}(C)$ $, 1 \leq i_1 < \cdots < i_r \leq m, 1 \leq j_1 < \cdots < j_r \leq m$, the minors (respectively the cofactors) of the matrix C with $i_1, i_2, \dots, i_r$ rows and $j_1, j_2, \dots, j_r$ columns. By definition

$$A_{12\dots m}^{12\dots m}\,(C) = M_{\aleph}^{\aleph}(C) = 1\ and\ M_{12\dots m}^{12\dots m}\,(C) = M_{\aleph}^{\aleph}(C) = det\,C.$$

Lemma (4.1.17)[123]: For the generalized characteristic polynomial $G_m(\lambda)\ of\ C \in Mat(m, \mathbb{C})\ and\ \lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{C}^m$ we have:

$$G_m(\lambda) = det\left(C + \sum_{k=1}^{m} \lambda_k E_{kk}\right)$$

$$= detC + \sum_{r=1<k<i1<i_2<\cdots<i_r<m}^{m-k} \sum \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_r} A_{k\,i_1\,i_2 \dots i_r}^{k\,i_1\,i_2 \dots i_r}(C) \tag{25}$$

Obviously $G_m(\lambda)$ is a polynomial in the variables

Lemma (4.1.18)[123]: For $C \in Mat(m, \mathbb{C})\ and\ \lambda \in \mathbb{C}^m$ we have

$$G_m(\lambda) = A_{\aleph}^{\aleph}\big(C_m(\lambda)\big) = detC_m(\lambda) = detC_m + \sum_{r=1}^{m} \lambda_r A_r^r\left(C_m\big(\lambda^{[r]}\big)\right), \tag{26}$$

$$A_k^k(C_m(\lambda)) = A_k^k(C_m) + \sum_{r=1, r\neq k}^{m} \lambda_r A_r^r\left(C_m\big(\lambda^{[r]}\big)\right), \tag{27}$$

$$G_m(\lambda) = A_{\aleph}^{\aleph}\big(C_m(\lambda)\big) = detC_{mm}(\lambda) detC_m + \sum_{r=1}^{m} \lambda_r A_r^r\left(C_m\big(\lambda^{[r]}\big)\right) \tag{28}$$

$$A_k^k(C_m(\lambda)) = A_k^k(C_m) + \sum_{r=1, r\neq k}^{m} \lambda_r A_{rk}^{rk}\left(C_m\big(\lambda^{[r]}\big)\right), \tag{29}$$

where for $\lambda \in C_m\ and\ 1 \leq k \leq m$ we have set

$$\lambda^{[k]} = (0, \dots, 0, \lambda_{k+1}, \dots, \lambda_m), \lambda^{\{k\}} = (\lambda_1, \lambda_2, \dots, \lambda_k, 0, \dots, 0). \tag{30}$$

Proof: We have for m = 2 using (25)

$$\begin{aligned} G_2(\lambda) &= detC_2 + \lambda_1 A_1^1(C_2) + \lambda_2 A_2^2(C_2) + \lambda_1\lambda_2 A_{12}^{12}(C_2) \\ &= detC_2 + \lambda_1[A_1^1(C_2) + \lambda_2 A_{12}^{12}(C_2)] + \lambda_2 A_2^2(C_2) \\ &= detC_2 + \lambda_1 A_1^1\left(C_2\big(\lambda^{[124]}\big)\right) + \lambda_2 A_2^2\left(C_2\big(\lambda^{[125]}\big)\right), \end{aligned}$$

$$G_2(\lambda) = detC_2 + \lambda_1 A_1^1(C_2) + \lambda_2[A_2^2(C_2) + \lambda_1 A_{12}^{12}(C_2)]$$
$$= detC_2 + \lambda_1 A_1^1\left(C_2\left(\lambda^{\{1\}}\right)\right) + \lambda_2 A_2^2\left(C_2\left(\lambda^{\{2\}}\right)\right).$$
For m =3 we have
$$G_3(\lambda) = detC_3 + \lambda_1 A_1^1(C_3) + \lambda_2 A_2^2(C_3) + \lambda_3 A_3^3(C_3) + \lambda_1\lambda_2 A_{12}^{12}(C_3) + \lambda_1\lambda_3 A_{13}^{13}(C_3)$$
$$+ \lambda_2\lambda_3 A_{23}^{23}(C_3) + \lambda_1\lambda_2\lambda_3 A_{123}^{123}(C_3)$$
$$= detC_2 + \lambda_1[A_1^1(C_3) + \lambda_2 A_{12}^{12}(C_3) + \lambda_3 A_{13}^{13}(C_3) + \lambda_2\lambda_3 A_{123}^{123}(C_3)]$$
$$+ \lambda_2[A_2^2(C_3) + \lambda_3 A_{23}^{23}(C_3)] + \lambda_3 A_3^3(C_3)$$
$$= detC_3 + \lambda_1 A_1^1\left(C_3\left(\lambda^{[124]}\right)\right) + \lambda_2 A_2^2\left(C_3\left(\lambda^{[125]}\right)\right) + \lambda_3 A_3^3\left(C_3\left(\lambda^{[126]}\right)\right),$$
$$G_3(\lambda) = detC_3 + \lambda_1 A_1^1(C_3) + \lambda_2[A_2^2(C_3) + \lambda_1 A_{12}^{12}(C_3)]$$
$$+ \lambda_3[A_1^1(C_3) + \lambda_1 A_{13}^{12}(C_3) + \lambda_2 A_{23}^{23}(C_3) + \lambda_1\lambda_2 A_{123}^{123}(C_3)]$$
$$= detC_3 + \lambda_1 A_1^1\left(C_3\left(\lambda^{\{1\}}\right)\right) + \lambda_2\left(A_2^2 C_3\left(\lambda^{\{2\}}\right)\right) + \lambda_3 A_3^3\left(C_3\left(\lambda^{\{3\}}\right)\right)$$
For m $>$ 3 the proof of (26) and (28) is the same. The identity (27) follows from (26) and (29) follows from (28).

The proof of Lemma (4.1.16) is based on Lemmas (4.1.19), (4.1.21) and (4.1.22) concerning the properties of a positive matrices.

Lemma (4.1.19)[123]: (Sylvester [159]) Let $C \in Mat(n,\mathbb{R})$ and $1 \le p < n$. We consider a matrix $B = (bik)_{p+1}^{n}$ defined by $b_{ik} = M_{12\ldots pk}^{12\ldots pi}(C)$. Then the following Sylvester determinant identity holds:
$$\det B = \left[M_{12\ldots p}^{12\ldots p}(C)\right]^{n-p-1} \det C.$$
Corollary (4.1.20)[123]: If $p = n - 2$ we have in particular
$$\begin{vmatrix} A_n^n(C) & A_{n-1}^n(C) \\ A_n^{n-1}(C) & A_{n-1}^{n-1}(C) \end{vmatrix} = A_{n-1n}^{n-1n}(C)A_{\aleph}^{\aleph}(C).$$
For arbitrary $1 \le p < q \le n$ we have
$$\begin{vmatrix} A_p^p(C) & A_q^p(C) \\ A_p^q(C) & A_q^q(C) \end{vmatrix} = A_{\aleph}^{\aleph}(C)A_{pq}^{pq}(C) \text{ or } \begin{vmatrix} A_p^p(C) & A_{pq}^{pq}(C) \\ A_{\aleph}^{\aleph}(C) & A_q^q(C) \end{vmatrix} = A_q^p(C)A_p^q(C). \quad (31)$$
Lemma (4.1.21)[123]: (Hadamard–Ficher's inequality [135], [136], see also [150]) For any positive definite matrix $C \in Mat(m,\mathbb{R}), m \in \mathbb{N}$, and any two subsets α and β with $\varnothing \subseteq \alpha$, $\beta \subseteq \{1,\ldots,m\}$ the following inequality holds:
$$\begin{vmatrix} M(\alpha) & M(\alpha \cup \beta) \\ M(\alpha \cap \beta) & M(\beta) \end{vmatrix} = \begin{vmatrix} A(\hat{\alpha}) & A(\hat{\alpha} \cup \hat{\beta}) & A(\hat{\beta}) \\ A(\hat{\alpha} \cup \hat{\beta}) & A(\hat{\beta}) & \end{vmatrix} \le 0, \quad (32)$$
where $M(\alpha) = M_\alpha^\alpha(C), A(\alpha) = A_\alpha^\alpha(C)$ and $\hat{\alpha} = \{1,\ldots,m\} \setminus \alpha$.

More precisely, see [135]; [136]. See also [150].

Let us set as before (see (31)) for $\lambda = (\lambda_1,\ldots,\lambda_k) \in \mathbb{C}^k$ *and* $C \in Mat(k,\mathbb{C})$
$$G_k(\lambda) = detC_k(\lambda), where\ C_k(\lambda) = C + \sum_{r=1}^{k} \lambda_r E_{rr}.$$
In the following lemma we use the notation for $\lambda = (\lambda_1,\ldots,\lambda_k) \in \mathbb{C}^k$:
$$\lambda^{]l[} = (\lambda_1,\ldots,\lambda_{l-1},0,\lambda_{l+1},\ldots,\lambda_k), 1 \le l \le k,$$

and $Gl(\lambda) = M_{12\ldots l}^{12\ldots l}\ (C_k(\lambda)), 1 \le l \le k$. For α and β such that $\emptyset \subseteq \alpha \subseteq \{1,2,\ldots,l\}$ and $\emptyset \subseteq \beta \subseteq \{l+1,\ldots,k\}$, with $l < k, C \in$ Mat $(k,\mathbb{C})$ we set

$$(A_\alpha^\alpha(C))_\beta^\beta := A_{\alpha\cup\beta}^{\alpha\cup\beta}(C), and\ G_l(\lambda)_\beta^\beta := \sum_{\emptyset\subseteq\alpha\{1,2,\ldots,l\}} \lambda_\alpha A_{\alpha\cup\beta}^{\alpha\cup\beta}(C).$$

By definition we have

$$G_l(\lambda) = A_{l+1\ldots k}^{l+1\ldots k}\ (C_k(\lambda)) = \left(A_\emptyset^\emptyset(C_k(\lambda))\right)_{l+1\ldots k}^{l+1\ldots k} = G_k(\lambda)_{l+1\ldots k}^{l+1\ldots k}. \tag{33}$$

Lemma (4.1.22)[123]: We have for $1 \le p \le l \le k$ and $C \in Mat(k,\mathbb{C})$

$$\frac{G_k(\lambda)}{G_l(\lambda)} = \frac{G_k(\lambda^{]p[}) + \lambda_p G_k(\lambda^{]p[})_p^p}{G_k(\lambda^{]p[})_{l+1\ldots k}^{l+1\ldots k} + \lambda_p G_k(\lambda^{]p[})_{pl+1\ldots k.}^{pl+1\ldots k}} \tag{34}$$

For the positive definite matrix C and $\lambda = (\lambda_1,\ldots,\lambda_k) \in \mathbb{R}^k$ with $\lambda_r \ge 0, r = 1,\ldots,k$, we have

$$(G_l(\lambda))^2\ \frac{\partial}{\partial\lambda_p}\frac{G_k(\lambda)}{G_l(\lambda)} = \begin{vmatrix} G_k(\lambda^{]p[})_p^p & G_k(\lambda^{]p[}) \\ G_k(\lambda^{]p[})_{pl+1\ldots k}^{pl+1\ldots k} & G_k(\lambda^{]p[})_{l+1\ldots k}^{l+1\ldots k}\end{vmatrix} \ge 0. \tag{35}$$

Proof: We have for $1 \le p \le l \le k$

$$\frac{\partial G_k(\lambda)}{\partial\lambda_p} = \frac{\partial}{\partial\lambda_p} det\left(C + \sum_{r=1}^k \lambda_r E_{rr}\right) = A_p^p\left(C\left(\lambda^{]p[}\right)\right) = G_k\left(\lambda^{]p[}\right)_p^p, so \tag{36}$$

$$G_k(\lambda) - \lambda_p G_k\left(\lambda^{]p[}\right)_p^p = G_k(\lambda)|_{\lambda_p=0} = G_k\left(\lambda^{]p[}\right),$$

hence

$$G_k(\lambda) = G_k\left(\lambda^{]p[}\right) + \lambda_p G_k\left(\lambda^{]p[}\right)_p^p, 1 \le p \le k.$$

Similarly, we have

$$G_l(\lambda) = G_l\left(\lambda^{]p[}\right) + \lambda p G_l\left(\lambda^{]p[}\right)_p^p\left(\lambda^{]p[}\right)_p^p = G_k\left(\lambda^{]p[}\right)_{l+1\ldots k}^{l+1\ldots k} + \lambda_p G_k \left(\lambda^{]p[}\right)_{pl+1\ldots k}^{pl+1\ldots k}, 1 \le p \le l.$$

Finally we get (34). Using the following formula:

$$\left(\frac{a + bx}{c + dx}\right)' = \frac{bc - ad}{(c + dx)^2}$$

we conclude that (34) implies the identity in (34).
To show the inequality in (34) we get

$$\begin{vmatrix} G_k(\lambda^{]p[})_p^p & G_k(\lambda^{]p[}) \\ G_k(\lambda^{]p[})_{pl+1\ldots k}^{pl+1\ldots k} & G_k(\lambda^{]p[})_{l+1\ldots k}^{l+1\ldots k}\end{vmatrix} = \begin{vmatrix} A_p^p(C_k(\lambda^{]p[})) & A_\emptyset^\emptyset(C_k(\lambda^{]p[})) \\ A_{pl+1\ldots k}^{pl+1\ldots k}(C_k(\lambda^{]p[})) & A_{l+1\ldots k}^{l+1\ldots k}(C_k(\lambda^{]p[}))\end{vmatrix}$$

$$= \begin{vmatrix} A_\alpha^\alpha(C) & A_{\alpha\cap\beta}^{\alpha\cap\beta}(C) \\ A_{\alpha\cup\beta}^{\alpha\cup\beta}(C) & A_\beta^\beta(C)\end{vmatrix} \ge 0,$$

where $C = C_k(\lambda^{]p[}), \alpha = \{p\}$ and $\beta = \{l+1, l+2,\ldots,k\}$.

Calculation of the matrix elements $\emptyset_p(t)$ for $t \in R^p$, their generalizations and Ξ_n^{pq}.

Let us recall (see (10) and (19)) that $\hat{\lambda}_k = \sum_{r=1}^{K-1} c_{rr}, 2 \leq r \leq m, \hat{\lambda}_1 = 0$ and

$$\Xi_n^{pq} = \max_{t\in\mathbb{R}^p} \left\| M\xi_n^{pq} (t)\right\|^2, 1 \leq p \leq q \leq m. \qquad (37)$$

To estimate

$$\max_{t\in\mathbb{R}^p} \left\| M\xi_n^{pq} (t)\right\|^2, = \max_{t\in\mathbb{R}^p} \left\| \xi_n^{pq} (t)1,1\right\|^2,,$$

where $\xi_n^{pq} (t) = iy_{qn}\, exp(\sum_{r=1}^{p} t_r \widetilde{A_{rn}})$ we shall find the exact formulas for the matrix elements

$$\emptyset_p(t) = \varphi_p^{(n)} (t) = T^{R,\mu_B^m}_{exp(\sum_{r=1}^{p} t_r E_{rn}1,1)}, t = (t_r)_{r=1}^{p} \in \mathbb{R}^p, 1 \leq p \leq m, \qquad (38)$$

of the restriction of the representation T^{R,μ_B^m} on the commutative subgroup $exp\left(\sum_{r=1}^{p} t_r E_{rn}\right) | t \in \mathbb{R}^p) \simeq \mathbb{R}^p$ of the group $B_0^{\mathbb{N}}$ and theirs generalization defined below. We note that

$$exp\left(\sum_{r=1}^{p} t_r E_{rn}\right) = I + \sum_{r=1}^{p} t_r E_{rn}.$$

For $1 \leq p \leq q, p, q \in \mathbb{N}$ we get

$$\xi_n^{pq} (t) = iy_{qn}\, exp\left(\sum_{r=1}^{p} t_r \widetilde{A_{rn}}\right) = iy_{qn}\, exp\, i\left[\sum_{r=1}^{p} tr\left(\sum_{k=1}^{r-1} x_{kr} ykn + y_{rn}\right)\right] \qquad (39)$$

we have used the expression $\widetilde{A_{rn}} = \sum_{k=1}^{r-1} x_{kr} ykn + y_{rn} = \sum_{k=1}^{r} x_{kr} y_{kn}$ (see (9)). We have

$$\tilde{T}^{R,\mu_B^m}_{exp(\sum_{r=1}^{p} t_r E_{rn})} = exp\left(\sum\nolimits_{r=1}^{p} t_r \widetilde{A_{rn}}\right) = exp\, i \sum_{r=1}^{p} t_r \left(\sum_{k=1}^{r} x_{kr} y_{kn}\right)$$

$$= exp\, i\left[\sum_{k=1}^{p} t_r \left(\sum_{r=k}^{p} x_{kr}\right) y_{kn}\right]$$

To obtain $\xi^{pp}(t)$ we generalize the function

$$\tilde{T}^{R,\mu_B^m}_{exp(\sum_{r=1}^{p} t_r E_{rn})}$$

in the following way. We replace in the latter identity the vectors $(t_r, \ldots, t_r) \in \mathbb{R}^{p-k+1}\, by (t_{rk})_{r=k}^{p} \in \mathbb{R}^{p-k+1}$ and denote the result by $\xi_{pp}(t)$: 8i

$$\xi_{pp}(t) = \xi_{pp}\left(\begin{bmatrix} t_{11} & \cdots & & \\ t_{21} & t_{22} & & \\ t_{31} & t_{32} & \ldots & \\ t_{p1} & t_{p2} & \ldots & t_{pp} \end{bmatrix}\right) = exp\, i\left[\sum_{k=1}^{p} t_r \left(\sum_{r=k+1}^{p} x_{kr} t_{rk} + t_{kk}\right) y_{kn}\right] \qquad (40)$$

To obtain $\xi^{pq}(t)$ we consider the function $\xi_{pq}(t\,;\, t_{qq}) = \xi_{pp}(t)\, exp(itqqyqn)$. We have

$$\xi_{pq}(t\,;\,t_{qq}) = \xi_{pp}\begin{pmatrix} t_{11} & \cdots & & & \\ t_{21} & t_{22} & & & \\ t_{31} & t_{32} & \ldots & & \\ t_{p1} & t_{p2} & \ldots & t_{pp} & t_{pp} \end{pmatrix}$$

$$:= exp\, i\left[\sum_{k=1}^{p}\left(\sum_{r=k+1}^{p} x_{kr}\, t_{rk} + t_{kk}\right) ykn + t_{qq} yq_{yn}\right].$$

Finally we have

$$\xi^{pp}(t) = \frac{\partial \xi_{pp}(t)}{\partial t_{pp}}\bigg|_{t_{kr}=tk\,,1\le r\le k\le p} \text{ and } \xi^{pq}(t) = \frac{\partial \xi_{pq}(t\,;\,t_{qq})}{\partial t_{qq}}\bigg|_{t_{qq}=0,t_{kr}=tk\,,1\le r\le k\le p}.$$

Let us define $\emptyset_p(t) = \int \xi_{pp}(t) d\mu(x,y), \emptyset_{pq}(t) = \int \xi_{pq}(t) d\mu(x,y)$, where $\mu(x,y) = \mu I\,(x) \otimes (\otimes_{n=m+1\;\mu C^{(n)}(y))}^{\infty}$ and $\mu_I\,(x)$ is the standard Gaussian measure in $\mathbb{R}\times\mathbb{R}^2 \times\cdots\times \mathbb{R}^m$.

Using definition (36) and the previous equalities we have finally

$$\Xi^{pp} = \max_{t\in\mathbb{R}^p}\left|\frac{\partial \emptyset_p x(t)}{\partial t_{pp}}\right|^2_{t_{kr}=t_k\,,1\le r\le k\le p},$$

$$\Xi^{pq} = \max_{t\in\mathbb{R}^p}\left|\frac{\partial \emptyset_p x(t)}{\partial t_{pp}}\right|^2_{t_{kr}=0,t_k\,,1\le r\le k\le p}. \tag{41}$$

Our aim is to estimate Ξ^{pq}. We shall use the notation $C_k := C_{\{1,2,\ldots,k\}}$ for Mat$(m,\mathbb{C})$ and $1 \le k \le m$ (see notation $C_\alpha\; for\; \phi \subseteq \alpha \subseteq \{1,\ldots,m\}$ in Lemma (4.1.8)).

Lemma (4.1.23)[123]: For $1 \le p \le q \le m$ and $\varphi pq(t) = \int \xi_{pq}(t) d\mu(x,y)$ we have

$$\emptyset_{pq}\begin{pmatrix} t_{11} & \cdots & & & & \\ t_{21} & t_{22} & & & & \\ t_{31} & t_{32} & t_{33} & & & \\ & & & \cdots & & \\ t_{p1} & t_{p2} & t_{p3} & \cdots & t_{pp;} & t_{pp} \end{pmatrix}$$

$$= \int_{\mathbb{R}^{\frac{(P-1)(P-2)}{2}+P}} exp\, i\left[\sum_{k=1}^{p}\left(\sum_{r=k+1}^{p} x_{kr}\, t_{rk}\right) ykn + t_{qq} y_{yn}\right] d\mu(x,y)$$

$$= \frac{1}{\sqrt{\det C_1(t)}} exp\left(-\frac{1}{2}\left[(CT,T) - (C_1(t)^{-1}d,d)\right]\right), \tag{42}$$

where we set $T = (t_{11},t_{22},t_{33},\ldots,t_{pp};\, t_{qq}) \in \mathbb{R}^{p+1}, C \in$ Mat$(p+1,\mathbb{C})$ is defined by

$$C := C_{p,q} := C_{\{1,2,\ldots,p,q\}} := \left(\begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1p} & c_{1q} \\ c_{12} & c_{22} & c_{23} & \cdots & c_{2p} & c_{2q} \\ c_{13} & c_{23} & c_{33} & \ldots & c_{3p} & c_{3q} \\ & & & \ddots & & \\ c_{1p} & c_{2p} & c_{3p} & \cdots & c_{pp} & c_{pq} \\ c_{1q} & c_{2q} & c_{3q} & \cdots & c_{pq} & c_{qq} \end{bmatrix}\right)$$

$$d = d_{21}(t), d_{31}(t), \ldots, d_{p1}(t);\; d_{32}(t), d_{42}(t), \ldots, dp2(t);\; \ldots;\; d_{pp-1}(t) \in \mathbb{R}^{\frac{(P-1)(P-2)}{2}},$$

$$d_{rs}(t) = t_{rs}e_s(t), 1 \le s < r < p, e_s(t) = (CT)_s$$

$$= \sum_{k=1}^{p} c_s k t_{kk} + c_{kk} x\, t_{kk}, 1 \le s \le p,$$

the operator

$$C_1(t) = 1 + C(t) \in Mat\left(\frac{(p-1)(p-2)}{2}, \mathbb{C}\right)$$

being defined by

$D(t)^{-1}C_1(t)D(t)^{-1}$

$$= \begin{pmatrix} c_{11} + t_{21}^{-2} & \dots & c_{11} & c_{12} & \dots & c_{12} & \dots & c_{1p-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{11} & \dots & c_{11} + t_{p1}^{-2} & c_{12} & \dots & c_{12} & \dots & c_{1p-1} \\ c_{12} & \dots & c_{12} & c_{22} + t_{32}^{-2} & \dots & c_{22} & \dots & c_{2p-1} \\ \dots & \vdots & \dots & \dots & \vdots & \dots & \vdots & \dots \\ c_{12} & \vdots & c_{12} & c_{22} & \vdots & c_{22} + t_{p^2}^{-2} & \vdots & c_{2p-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{1p-1} & \dots & c_{1p-1} & c_{2p-1} & \dots & c_{2p-1} & \dots & c_{p-1p-1} + t_{pp-1}^{-2} \end{pmatrix} \quad (43)$$

where $D(tz = diag(t_{21}, \dots, t_{p1};\ t_{32}, \dots, t_{p2p2};\ t_{43}, \dots, t_{p3p3};\ \dots;\ t_{pp-1})$. We have

$$detC_1(t) = 1 + \sum_{r=1 1 \le i_1 < i_{22} < \dots < i_r \le p}^{p} \sum \alpha_{i_1}^2 \alpha_{i_2}^2 \dots \alpha_{i_r}^2 M_{i_1 i_2 \dots i_r}^{i_1 i_2 \dots i_r}(C_p) = \sum_{s=k+1}^{p} t_{sk}^2. \quad (44)$$

Lemma (4.1.24)[123]: For $1 \le p \le q \le$m we have

$$\Xi^{pq} \ge \Psi^{pq}, \quad (45)$$

where

$$\Psi^{pq} = \frac{(M_{12\dots p-1q}^{12\dots p-1p}(C_{p,q}))^2 \, exp(-1)}{(M_{12\dots p-1}^{12\dots p-1}(C_p) M_{12\dots p}^{12\dots p}(C_p) + \sum_{k=2}^{p} \widehat{\lambda}_k (A_k^p(C_p))^2}. \quad (46)$$

List of formulas for Ψ^{pq} for small p and $p < q$.

$$\Psi^{11} = c_{11}\, exp(-1), \Psi^{1q} = \frac{c_{1q}^2 \, exp(-1)}{c_{11}}, 1 \le q, \quad (47)$$

$$\Psi^{22} = \frac{(M_{12}^{12})^2 \, exp(-1)}{c_{11}(M_{12}^{12} + c_{11}^2)}, \Psi^{2q} = \frac{(M_{1q}^{12})^2 \, exp(-1)}{c_{11}(M_{12}^{12} + c_{11}^2)}, 2 \le q, \quad (48)$$

$$\Psi^{3q} = \frac{(M_{12q}^{123})^2 \, exp(-1)}{M_{12}^{12} M_{123}^{123} + c_{11}(M_{13}^{12})^2 + (c_{11} + c_{22})(M_{12}^{12})^2}, 3 \le q, \quad (49)$$

$$\Psi^{4q} = \frac{(M_{123q}^{1234})^2 \, exp(-1)}{M_{123}^{123} M_{1234}^{1234} + c_{11}(M_{124}^{123})^2 + (c_{11} + c_{22})(M_{124}^{123})^2 + (c_{11} + c_{22} + c_{33})(M_{123}^{123})^2.} \quad (50)$$

Proof of Lemmas (4.1.23) and (4.1.24). For a positive definite operator C in the space Rm we have the well-known formulas:

$$1\sqrt{(2\pi)^m} \int_{\mathbb{R}^m} exp\left(-\frac{1}{2}(Cx, x)\right) dx = \frac{1}{\sqrt{detC}}. \quad (51)$$

Using formula (50) we obtain the following formula for $d \in \mathbb{R}^m$:

$$\frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} exp\left(-\frac{1}{2}(Cx,x)+ (d,x)\right)dx = \frac{1}{\sqrt{detC}} exp\frac{(C^{-1}d,d)}{2}, \qquad (52)$$

and as a particular case for m =1 we have

$$\frac{1}{\sqrt{2\pi}} R\int_{\mathbb{R}} exp\left(-\frac{1}{2}cx^2 + dx\right)dx = \frac{1}{\sqrt{c}} exp\left(\frac{d^2}{2^c}\right). \qquad (53)$$

To obtain (52) from (51) we use the following formula:

$$(Cx,x)-2(d,x)=C(x - x_0),(x - x_0)_{(C^{-1}d,d)}, \text{where } x_0 = C^{-1}d. \qquad (54)$$

Indeed we find $x_0 \in \mathbb{R}^m$ and $d_0 \in \mathbb{R}$ such that

$$(Cx,x)- 2(d,x) = (C(x - x_0),(x - x_0)) + d_0.$$

We have

$$(Cx,x)- 2(d,x) = (C(x - x_0),(x - x_0)) + d_0$$
$$= (Cx,x) - 2(Cx_0,x)+ (Cx_0,x_0)+d_0,$$

so $Cx_0 = d$ hence $x_0 = C^{-1}$ d and since $(Cx_0,x_0)+d_0 = 0$ we conclude that $d_0 = -(Cx_0,x_0) = -(CC^{-1}d,C^{-1}d) = -(C^{-1}d,d)$.

Fourier transform for the Gaussian measure μC in the space $\mathbb{R}^m$ is:

$$\frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbb{R}^m} exp\, i(y,x)d\mu_C(x) = exp\left(-\frac{1}{2}(Cy,y)\right), y \in \mathbb{R}^m.$$

Let p = 1. Using (51)–(53) we have

$$\emptyset_1(t_{11}) = \int_{\mathbb{R}} exp(it_{11}y_{1n})d\mu(y) = exp\left(-\frac{1}{2}c_{11}t_{11}^2\right);$$

$$\emptyset_{1q}(t_{11};\, t_{qq}) = \int_{\mathbb{R}^2} exp\, i(t_{11}y_{1n} + t_{qq}y_{qn})d\mu(y)$$
$$= exp\left(-\frac{1}{2}c_{11}t_{11}^2 + 2c_{1q}\,t_{11}t_{qq} + c_{qq}t_{qq}^2\right);$$

$$M\xi^{1q}(t_{11}) = \int_{\mathbb{R}} iy_{qn}\, exp(it_{11}y_{1n})d\mu(y) = \left.\frac{\partial\emptyset_{1q}(t_{11};\, t_{qq})}{\partial t_{qq}}\right|_{t_{qq}=0}$$
$$= -c_{1q}\, t_{11} exp\left(-\frac{1}{2}c_{11}^2t_{11}^2\right), |M\xi^{1,q}(t_{11})|^2 = c_{1q}^2\, t_{11}^2\, exp(-c_{11}t_{11}^2);$$

$$\Xi^{1q} = \max_{t_{11}\in\mathbb{R}}|M\xi^{1q}(t_{11})|^2 = \frac{c_{1q}^2\ exp(-1)}{c_{11}} = \Psi^{1q},$$

we have used the obvious result max

$$\max_{x\in\mathbb{R}} f(x) = f\left(\frac{1}{a}\right) = \frac{1}{ea}, where\ f(x) = x\, exp(-ax), a > 0. \qquad (55)$$

This shows (45) for (p, q) = (1, q).

To show (42) in the general case we note that

$$\sum_{k=1}^{p}\left(\sum_{r=k+1}^{p} x_{kr}\, t_{rk} \;+\; t_{kk}\right) y_{kn} \;+\; t_{qq} y_{qn} \;=(a(x)\;+T,y)\mathbb{R}^{p+1},$$

where

$y \;=\; (y_{1n}, y_{2n}, \ldots, y_{pn};\; y_{qn}), T \;=\; (t_{11}, t_{22}, \ldots, t_{pp};\; t_{qq}) \;\in\; \mathbb{R}^{p+1},$

$$a(x)\;=\left(a_1(x), a_2(x), \ldots, a_p(x);\;0\right) \in\; \mathbb{R}^{p+1}, a_k(x)\;=\;\sum_{r=k+1}^{p} x_{kr}\, t_{rk}\;=\;(xt)_{kk}\,,$$

$$x\;=\;\sum_{1<k<r\leq m} x_{kr} E_{kr}\,, t=\sum_{1<r<k\leq m} t_{kr} E_{kr}\,, 1\;\leq\;k\;\leq\;p.$$

Using the definition of the Fourier transform we have

$$\emptyset_{pq}(t\,;\,t_{qq})\;=\int\;\int_{\mathbb{R}^{p+1}} exp\; i\left[\sum_{k=1}^{p}\left(\sum_{r=k}^{p} xk_r\, t_{rk}\right) y_{kn}\;+t_{qq} y_{qn}\right] d\mu(x,y)$$

$$=\int exp\; i((x)+T,y) d\mu(x,y)\;=\int exp\left[-\frac{1}{2}(Ca(x)+T), a(x)+\;T)\right] d\mu I\;(x)\,.$$

Since

$$(C(a(x)+T), a(x)+\;T)=(Ca(x), a(x))+2(a(x), CT)+\;(CT\;,T\;),$$

we have

$$\emptyset_{pq}\left(t\,;\,t_{qq}\right)=exp\left[-\frac{1}{2}(CT\;,T\;)\right]\int exp\left(-\frac{1}{2}\left[\left(Ca(x), a(x)\right)\right.\right.$$
$$\left.\left.+\,2(a(x), CT)\right]\right) d\mu_I\;(x). \qquad (56)$$

To calculate the latter integral we use (52). Let us introduce the notation

$$X\;=\;(x_{12};\;x_{13}, x_{23};\;\ldots;\;x_{1p}, \ldots;\;x_{p-1}p)\;\in\mathbb{R}^{\frac{(p-1)(p-2)}{2}}.$$

We show that

$$(Ca(x), a(x))+2(a(x), CT)=(C(t)X, X)+2(d(t), X)$$

for some

$$d(t)\;\in\;\mathbb{R}^{\frac{(p-1)(p-2)}{2}}\; and\; C(t)\;\in\;Mat\left(\frac{(p-1)(p-2)}{2}, \mathbb{R}\right).$$

We have

$$(a(x), CT)=\sum_{k=1}^{p} a_k(x)(CT\;)_k\;=\sum_{k=1}^{p}\sum_{r=k+1}^{p} x_{kr}\, t_{rk} e_k(t)\;=\sum_{1\leq k<r\leq p} x_{kr}\, t_{rk} e_k(t)$$
$$=\sum_{1\leq k<r\leq p} x_{kr} d_{rk}(t)\;=(X, d(t)),$$

where

$$d(t)\;=(d_{rk}(t))_{1\leq k<r\leq p}\in\;\mathbb{R}^{\frac{(p-1)(p-2)}{2}},$$

$$d_{rk}(t) = t_{rk}e_k(t)\ and\ e_k(t) = (CT)_k = \sum_{r=1}^{p} c_{kr}\, t_{rr} + c_{kq}\, t_{qq}, 1 \le k \le p-1.$$

Further

$$(Ca(x), a(x)) = \sum_{1\le k,n\le p} c_{kn}a_k(x)a_n(x) = \sum_{1\le k,n\le p} c_{kn} \sum_{r=k+1}^{p} x_{kr}t_{rk} \sum_{s=n+1}^{p} x_{ns}\, t_{sn}$$
$$= \sum_{1\le k<r\le p}\ \sum_{1\le n<s\le p} c_{kn}t_{rk}t_{sn}x_{kr}x_{ns} = (C(t)X, X),$$

where the operator C(t) is defined by its entries:

$$(C(t))_{kr,ns} = c_{kn}t_{rk}t_{sn}\ for\ 1 \le k < r \le p and\ 1 \le n < s \le p. \quad (57)$$

This show the representation (43) for the operator $C_1(t)$. Finally we have

$$(Ca(x), a(x)) = (C(t)X, X) \text{and } (a(x), CT) = (X, d(t)).$$

Putting the latter equalities in (56) we get using (52)

$$\emptyset_{pq}(t\,;\,t_{qq}) = exp\left[-\frac{1}{2}(CT, T)\right] \int exp\left(-\frac{1}{2}[(C(t)X, X) + 2(X, d(t))]\right) d\mu_I\,(x)$$
$$= \frac{1}{\sqrt{detC_1(t)}}\, exp\left(-\frac{1}{2}[(CT, T) - (C_1(t)^{-1}d(t), d(t))]\right),$$

where $C_1(t) = I + C(t)$. This shows (44) of Lemma (4.1.23).

We estimate now $\varXi^{pq}$. For (p, q) = (2, 2) we get

$$\emptyset_2(t) = \frac{1}{\sqrt{detC_1(t)}} exp\left(-\frac{1}{2}[(CT, T) - (C_1(t)^{-1}d(t), d(t))]\right)$$
$$= \frac{1}{\sqrt{1 + c_{11}t_{21}^2}} \exp\left(-\frac{1}{2}\left[c_{11}t_{11}^2 + 2c_{12}t_{11}t_{22} + c_{22}t_{22}^2 - \frac{(c_{11}t_{11} + c_{12}t_{22})^2 t_{21}^2}{1 + c_{11}t_{21}^2}\right]\right),$$

where

$$T = (t_{11}, t_{22}), d(t) = d_{21}(t) = t_{21}e_1(t) = t_{21}(c_{11}t_{11} + c_{12}t_{22}),$$
$$e_1(t) = c_{11}t_{11} + c_{12}t_{22}, e_2(t) = c_{21}t_{11} + c_{22}t_{22},$$
$$C = C_2 = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22}\end{pmatrix}, C(t) \overset{(61)}{=} c_{11}t_{21}^2, C_1(t) = 1 + c_{11}t_{21}^2,$$
$$\frac{\partial\emptyset_2(t)}{\partial t_{11}} = \left[-(c_{11}t_{11} + c_{12}t_{22}) + \frac{(c_{11}t_{11} + c_{12}t_{22})c_{11}t_{21}^2}{1 + c_{11}t_{21}^2}\right]$$
$$\times \frac{exp(-\frac{1}{2}[(CT, T) - (C_1(t)^{-1}d(t), d(t))])}{\sqrt{detC_1(t)}},$$
$$\frac{\partial\emptyset_2(t)}{\partial t_{22}} = \left[-(c_{21}t_{11} + c_{22}t_{22}) + \frac{(c_{11}t_{11} + c_{12}t_{22})c_{12}t_{21}^2}{1 + c_{11}t_{21}^2}\right]$$
$$\times \frac{exp(-\frac{1}{2}[(CT, T) - (C_1(t)^{-1}d(t), d(t))])}{\sqrt{detC_1(t)}}.$$

Let $e_1(t) = c_{11}t_{11} + c_{12}t_{22} = 0\ so\ t_{11} = -c_{12}t_{22}/c_{11}$. In this case

$$(CT\ ,T\ )\ =\ c_{11}t_{11}^2+2c_{12}t_{11}t_{22}\ +c_{22}t_{22}^2=\left(\frac{c_{12}^2}{c_{11}}-2\frac{c_{12}^2}{c_{11}}+\ c_{22}\right)t_{22}^2=\ \frac{M_{12}^{12}}{c_{11}}t_{22}^2,$$

$$c_{12}t_{11}+c_{22}t_{22}\ =-\left(\frac{c_{12}c_{12}}{c_{11}}+c_{22}\right)t_{22}\ =\ \frac{c_{11}c_{22}\ -c_{12}^2}{c_{11}}=\frac{M_{12}^{12}}{c_{11}}.$$

Finally

$$|M\xi^{22}(t)|^2\ =|Miy_{2n}\,exp(it_{11}\ +it_{22}(x_{12}y_{1n}\ +y_{2n}))|^2\ =\left|\frac{\partial\emptyset_2(t)}{\partial t_{22}}\right|^2_{e_1(t)=0,t_{21}=t_{22}}$$

$$=\frac{\left(\frac{M_{12}^{12}}{c_{11}}t_{22}\right)^2\ exp\left(\frac{M_{12}^{12}}{c_{11}}t_{22}^2\right)}{1\ +c_{11}t_{22}^2}\geq\ \frac{M_{12}^{12}}{c_{11}}t_{22}^2\,exp\left[-\left(\frac{M_{12}^{12}}{c_{11}}c_{11}\right)t_{22}^2\right].$$

We have used the inequality

$$1\ +x\leq\ exp\,x,x\ \in\ \mathbb{R}. \tag{58}$$

Hence if we denote $t\ =\ (t_{11},t_{22})\ \in\ \mathbb{R}^2$ we have using (41)

$$\Xi^{22}\ =\ \min_{t\in\mathbb{R}^2}|M\xi^{22}(t|^2)>\Psi^{22}:=\ \frac{(M_{12}^{12}\ )^2\,exp(-1)}{c_{11}(M_{12}^{12}+c_{11}^2)}.$$

This shows (45) for (p, q) = (2, 2). For (2, q), 2<q, we have

$$\emptyset_{2q}\begin{pmatrix}t_{11} & & \\ t_{21} & t_{22;} & t_{qq}\end{pmatrix}\ =\int_{\mathbb{R}^{1+3}}exp\,i\left[t_{11}y_{1n}\ +(t_{21}x_{12}y_{1n}\ +t_{22}y_{2n})+t_{qq}y_{qn}\right]d\mu(x,y)$$

$$=\ \frac{1}{\sqrt{1\ +c_{11}t_{21}^2}}exp-\frac{1}{2}(c_{11}t_{11}^2+c_{22}t_{22}^2+c_{qq}t_{qq}^2+2c_{12}t_{11}t_{22}$$

$$+2c_{1q}\ t_{11}t_{qq}\ +\ 2c_{2q}\ t_{22}t_{qq}\ -\frac{(c_{11}t_{11}\ +c_{12}t_{22}\ +c_{1q}\ t_{qq})^2t_{21}^2}{1\ +c_{11}t_{21}^2}$$

$$=\ \frac{1}{\sqrt{detC_1(t)}}\ exp\left(-\frac{1}{2}[(CT\ ,T\ )-(C_1(t)^{-1}d(t),d(t))]\right),$$

where

$T=(t_{11},t_{22};\ t_{qq})\ \in\ \mathbb{R}^3,d(t)\ =t_{21}(c_{11}t_{11}\ +c_{12}t_{22}\ +c_{1q}\ t_{qq})\ =:\ t_{21}e_1(t)\ \in\mathbb{R},$

$$e_1(t)\ =\ c_{11}t_{11}\ +c_{12}t_{22}\ +c_{1q}\ t_{qq},e_2(t)\ =\ c_{21}t_{11}\ +\ c_{22}t_{22}\ +\ c_{2q}\ t_{qq},$$

$$C\ =\ C_{2,q}=\begin{pmatrix}c_{11} & c_{12} & c_{1q}\\ c_{12} & c_{22} & c_{2q}\\ c_{1q} & c_{2q} & c_{qq}\end{pmatrix}\ ,C_1(t)\ =\ detC_1(t)\ =\ 1+c_{11}t_{21}^2\ ,$$

$$\left.\frac{\partial\emptyset_{2q}\ (t\ ;\ t_{qq}\ )}{\partial t_{qq}}\right|_{t_{qq}=0}$$

$$=\left[(-\left(c_{1q}\ t_{11}\ +c_{2q}\ t_{22}\ +c_{qq}t_{qq}\right)+\frac{(c_{11}t_{11}\ +c_{12}t_{22}\ +\ c_{1q}\ t_{qq})c_{1q}\ t_{21}^2}{1+c_{11}t_{21}^2)}\right]$$

$$\times exp\left(-\frac{1}{2}[(CT\ ,T\ )-(C_1(t)^{-1}d,d)]\right)\frac{1}{\sqrt{detC_1(t)}}\ ,$$

$$\left.\frac{\partial \emptyset_{2q}\ (t\ ;\ t_{qq})}{\partial t_{qq}}\right|_{t_{qq}=0} = \left[(-(c_{1q}\ t_{11}\ + c_{2q}\ t_{22}\ ) + \frac{(c_{11}t_{11}\ + c_{12}t_{22}\ )c_{1q}\ t_{21}^2}{1 + c_{11}t_{21}^2)}\right]$$

$$\left.\times exp\left(-\frac{1}{2}(CT\ ,T\ )\right)\frac{1}{detC_1(t)}\right|_{t_{qq}=0}\ .$$

Let $t_{qq} = 0$. We chose d(t) = 0 so we have $c_{11}t_{11}\ + c_{12}t_{22}\ =\ 0$ and $t_{11}\ =\ -\frac{c_{12}t_{22}}{c_{11}}$. In this case

$$(CT\ ,T\ )\ =\ c_{11}t_{11}^2 + 2c_{12}t_{11}t_{22}\ + c_{22}t_{22}^2 = \left(\frac{c_{12}^2}{c_{11}} - 2\frac{c_{12}^2}{c_{11}} + c_{22}\right)t_{22}^2 =\ \frac{M_{12}^{12}}{c_{11}}t_{22}^2,$$

$$c_{1q}\ t_{11}\ +\ c_{2q}\ t_{22}\ = \left(-\frac{c_{12}c_{1q}}{c_{11}} + c_{2q}\right)t_{22}\ =\ \frac{c_{11}c_{2q}\ -\ c_{12}c_{1q}}{c_{11}}t_{22}\ = \frac{M_{12}^{12}}{c_{11}}t_{22}.$$

Finally, if we denote $t\ =\ (t11, t22)\ \in \mathbb{R}^2$, we have

$$|M\xi^{2q}\ (t)|^2\ = Miy_{qn}\ exp|it_{11}\ + it_{22}(x_{12}y_{\ 1n} + y_{2n})|^2\ = \left|\frac{\partial \emptyset_{2q}\ (t\ ;\ t_{qq})}{\partial t_{qq}}\right|^2_{t_{qq}=0,e1(t)=0}$$

$$= \frac{\left(\frac{M_{1q}^{12}}{c_{11}}t_{22}\right)^2\ \ exp - \left(\frac{M_{12}^{12}}{c_{11}}t_{12}^2\right)}{1 + c_{11}t_{22}^2} \overset{(62)}{>} \left(\frac{M_{1q}^{12}}{c_{11}}t_{22}\right)^2 exp\left(-\left(\frac{M_{12}^{12}}{c_{11}} + c_{11}\right)t_{22}^2\right).$$

By (55) we conclude using (41) that

$$\Xi^{2q}\ =\ \max_{t\in\mathbb{R}^2}|M\xi 2q\ (t)|^2 \geq\ \max_{t_{22}\in\mathbb{R}}\left|\frac{\partial \emptyset_{2q}\ (t\ ;\ t_{qq})}{\partial t_{qq}}\right|^2_{t_{qq}=0,e_1(t)=0} \geq \frac{(M_{1q}^{12}\ )^2\ exp(-1)}{c_{11}(M_{12}^{12} + c_{11}^2)} =\ \Psi^{2q}\ .$$

This shows (45) for $(p, q)\ =\ (2, q), 2 < q$.

For n =3 we have

$$\emptyset_3\begin{pmatrix} t_{11} & & \\ t_{21} & t_{22} & \\ t_{31} & t_{32} & t_{33} \end{pmatrix} = \frac{1}{\sqrt{detC_1(t)}}\ exp\left(-\frac{1}{2}[(CT\ ,T\ ) - (C_1(t)^{-1}d, d)]\right),$$

where

$$T\ =\ (t_{11}, t_{22}, t_{33}), d(t)\ = (\ d_{21}(t\ ), d_{31}(t\ ), d_{32}(t)),$$
$$d_{21}(t)\ =\ t_{21}e_1(t\ ), d_{31}(t)\ =\ t_{31}e_1(t\ ), d_{32}(t)\ =\ t_{32}e_2(t\ ),$$
$$e_1(t)\ =\ c_{11}t_{11}\ + c_{12}t_{22}\ +\ c_{13}t_{33}, e_2(t)\ = c_{21}t_{11}\ +\ c_{22}t_{22}\ + c_{23}t_{33},$$

$$C\ =\ C_3\ = \begin{pmatrix} t_{11} & c_{12} & c_{13} \\ t_{12} & c_{22} & c_{23} \\ t_{13} & c_{23} & c_{33} \end{pmatrix}, C(t) \overset{(61)}{=} \begin{pmatrix} c_{11}t_{21}^2 & c_{11}t_{21}t_{31} & c_{12}t_{21}t_{32} \\ c_{11}t_{21}t_{31} & c_{11}t_{31}^2 & c_{12}t_{31}t_{32} \\ c_{12}t_{21}t_{32} & c_{12}t_{31}t_{32} & c_{22}t_{32}^2 \end{pmatrix},$$

hence

$$C_1(t)\ =\ I\ +\ C(t)\ = \begin{pmatrix} 1 + c_{11}t_{21}^2 & c_{11}t_{21}t_{31} & c_{12}t_{21}t_{32} \\ c_{11}t_{21}t_{31} & 1 + c_{11}t_{31}^2 & c_{12}t_{31}t_{32} \\ c_{12}t_{21}t_{32} & c_{12}t_{31}t_{32} & 1 + c_{22}t_{32}^2 \end{pmatrix}$$

$$= \ diag(t_{21}, t_{31}, t_{32}) \begin{pmatrix} t_{11} + t_{21}^{-2} & c_{11} & c_{12} \\ t_{11} & c_{11}t_{31}^{-2} & c_{12} \\ t_{12} & c_{12} & c_{22} + t_{32}^{-2} \end{pmatrix}, diag(t_{21}, t_{31}, t_{32}).$$

We show the following inequality for an operator C of order n such that I +C >0:

$$det(I \ + C) \ \leq \ exp \, tr \, C. \tag{59}$$

Indeed by Hadamard inequality (see [130] or [137]) we have for positive operator C of order n

$$detC \leq \prod_{i=1}^{n} c_{ii} \ .$$

Using the Hadamard inequality and (58) we have for an operator C such that I +C >0

$$det(I \ + C) \ \leq \prod_{i=1}^{n} (1 \ + c_{ii}) \overset{(58)}{\leq} \prod_{i=1}^{n} exp \, c_{ii} \ = \ exp\left(\sum_{i=1}^{n} c_{ii}\right) = \ exp(trC),$$

where we denote by tr$C$ the trace of an operator C in the space C_n. Using (59) and (57) we conclude that

$$\det\big(I \ + C(t)\big) \leq \ trC(t) = \ exp\left[\sum_{k=1}^{p-1} c_{kk} \, (\sum_{r=k+1}^{p} t_{rk}^2)\right] = \ exp\left(\sum_{k=1}^{p-1} c_{kk} \, \alpha_k^2\right), \tag{60}$$

where $\alpha_k^2 = \sum_{r=k+1}^{p} t_{rk}^2$ since by (57) we have

$$trC(t) \ = \sum_{1 \leq k < r \leq p} C(t)_{kr,kr} \ = \sum_{1 \leq k < r \leq p} c_{kk} t_{rk}^2 = \sum_{k=1}^{p-1} c_{kk} \sum_{r=k+1}^{p} t_{rk}^2. \tag{61}$$

Using (25) we get

$$detC_1(t) \ = \ t_{21}^2 t_{31}^2 t_{32}^2 (detB \ + \ \lambda_1 A_1^1 + \ \lambda_1 A_2^2 + \lambda_3 A_3^3 + \lambda_1 \lambda_2 A_{12}^{12} + \lambda_1 \lambda_3 A_{13}^{13} + \ \lambda_2 \lambda_3 A_{23}^{23} + \lambda_1 \lambda_2 \lambda_3 A_{123}^{123})$$

$$= \ t_{21}^2 t_{31}^2 t_{32}^2 [\begin{vmatrix} c_{11} & c_{11} & c_{12} \\ c_{11} & c_{11} & c_{22} \\ c_{12} & c_{12} & c_{22} \end{vmatrix} + \left(\frac{1}{t_{21}^2} + \frac{1}{t_{31}^2}\right) \begin{vmatrix} c_{11} & c_{22} \\ c_{12} & c_{22} \end{vmatrix}$$

$$+ \ \frac{1}{t_{21}^2 t_{31}^2} c_{22} \ + \left(\frac{1}{t_{21}^2 t_{32}^2} + \frac{1}{t_{31}^2 t_{32}^2}\right) c_{11} \ + \ \frac{1}{t_{21}^2 t_{31}^2 t_{32}^2}]$$

$$= \ 1 \ + c_{11}(t_{21}^2 + t_{31}^2) + c_{22} t_{32}^2 + M_{12}^{12}(t_{21}^2 + t_{31}^2) t_{32}^2.$$

Finally we have

$$detC_1(t) = \ 1 \ + c_{11}\alpha_1^2 + c_{22}\alpha_2^2 + M_{12}^{12}\alpha_1^2\alpha_2^2, \qquad \text{where } \ \alpha_1^2 = \ t_{21}^2 + \ t_{31}^2, \alpha_2^2 = \ t_{32}^2.$$

For general n we have by analogy (it shows thus (44))

$$detC_1(t) \ = \ 1 + \sum_{r=1\_1 \leq i_1 < i_2 < \cdots < i_r \leq n-1}^{n-1} \alpha_{i_1}^2 \alpha_{i_2}^2 \ldots \alpha_{i_r}^2 M_{i_1 i_2 \ldots i_r}^{i_1 i_2 \ldots i_r} (C_n), where \ \alpha_k^2 = \sum_{s=k+1}^{n} t_{sk}^2.$$

For n =3 we have

$$\frac{\partial \emptyset_3(t)}{\partial t_{33}} = \left[-\frac{1}{2}\frac{\partial (CT,T\ )}{\partial t_{33}} + \frac{\partial\left(C_1(t)^{-1}d(t),d(t)\right)}{\partial t_{33}}\right]\frac{exp(-\frac{1}{2}[(CT\ ,T\ )-(C_1(t)^{-1}d(t),d(t))])}{\sqrt{detC_1(t)}},$$

$$\frac{\partial \emptyset_3(t)}{\partial t_{33}} = \left(-e3(t) + \frac{\partial\left(C_1(t)^{-1}d(t),d(t)\right)}{\partial t_{33}}\right)\frac{exp(-\frac{1}{2}[(CT\ ,T\ )-(C_1(t)^{-1}d(t),d(t))])}{\sqrt{detC_1(t)}}detC_1(t).$$

We calculate $|\partial\emptyset_3(t)/\partial t_{33}|^2$ under the conditions $e_1(t) = e_2(t) = 0$ on the variables $t = (t_{11}, t_{22}, t_{33}) \in \mathbb{R}^3$. It gives us

$$\begin{cases} c_{11}t_{11} + c_{12}t_{22} + c_{13}t_{33} = 0, \\ c_{21}t_{11} + c_{22}t_{22} + c_{23}t_{33} = 0. \end{cases}$$

The solutions are

$$t_{11} = \frac{M_{23}^{12}(C_3)}{M_{12}^{12}(C_3)}t_{33} = \frac{A_1^3(C_3)}{A_3^3(C_3)}t_{33}, t_{22} = -\frac{M_{13}^{12}(C_3)}{M_{12}^{12}(C_3)}t_{33} = \frac{A_2^3(C_3)}{A_3^3(C_3)}t_{33}. \quad (62)$$

In general, for the matrix C_n conditions $e_1(t) = e_2(t) =\cdot\cdot\cdot= e_{n-1}(t) = 0$ gives us the system

$$\begin{cases} c_{11}t_{11} + c_{12}t_{22} +\cdot\cdot\cdot+ c_{1n}t_{nn} = 0, \\ c_{21}t_{11} + c_{22}t_{22} +\cdot\cdot\cdot+ c_{2n}t_{nn} = 0, \\ \vdots \\ c_{n-11}t_{11} + c_{n-12}t_{22} +\cdot\cdot\cdot+ c_{n-1n}t_{nn} = 0 \end{cases} \quad (63)$$

and the following solutions:

$$t_{kk} = (-1)^{k+n}\frac{M_{12\ldots k-1k+1\ldots n}^{12\ldots k-1kk+1\ldots n-1}(C_n)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)}t_{nn} = \frac{A_k^n(C_n)}{A_n^n(C_n)}t_{nn}, 1 \le k \le n-1. \quad (64)$$

If we denote $e_k(t) = \sum_{r=1}^{n} c_{kr}\, t_{rr}$ we get

$$(CT,T) = \sum_{1\le k,r\le n} c_{kr}\, t_{rr}\, t_{kk} = \sum_{k=1}^{n} e_k(t\ )t_{kk}, \frac{1}{2}\frac{\partial (CT,T\ )}{\partial t_{nn}} = e_n(t\ ). \quad (65)$$

Under conditions (63) we have

$$e_n(t) = \sum_{r=1}^{n} c_{nr}\frac{A_r^n(C_n)}{A_n^n(C_n)}t_{nn} = \frac{M_{12\ldots n}^{12\ldots n}(C_n)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)}t_{nn}, \frac{\partial(C_1(t)^{-1}d(t),d(t))}{\partial t_{nn}} = 0 \quad (66)$$

and

$$(CT\ ,T \overset{(65)}{=} \sum_{k=1}^{n} e_k(t\ )t_{kk} = en(t\ )t_{nn} = \frac{M_{12\ldots n}^{12\ldots n}(C_n)}{M_{12\ldots n-1}^{12\ldots n-1}(C_n)}t_{nn}^2. \quad (67)$$

For n = 3 using (66) and (67) we can calculate

$$e_3(t) = \frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}t_{33}, (CT,T) = \frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}t_{33}^2, \frac{\partial(C_1(t)^{-1}d(t),d(t))}{\partial t_{33}} = 0.$$

If, in addition, $e_1(t) = e_2(t) = 0$, we have (see (62))

$trC(t) \;=\; c_{11}(t_{22}^2 + t_{33}^2) + \; c_{22}t_{33}^2 = \left[c_{11}\left(\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}\right)^2 + \;1\right) + c_{22})\right] t_{33}^2.$

For n =3 we have if e1$(t) \;=\; e_2(t) \;=\; 0$, using the values for $t_{22}, e_3(t)$ and (CT , T)

$$\left|\frac{\partial \emptyset_3(t)}{\partial t_{33}}\right|^2 \;=\; \frac{e_3^2(t)\; exp(-(CT\;,T\;))}{detC_1(t)} \overset{(64)}{\geq} e_3^2(t)\; exp(-(CT\;,T\;) - \; trC(t))$$

$$= \left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}\right)^2 t_{33}^2\; exp\left[-t_{33}^2\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)} + (c_{11} + c_{22}) \;+\; c_{11}\;\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}\right)^2\right)\right].$$

We get by (55)

$$\max_{t_{33}\in\mathbb{R}}\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}\right)^2 t_{33}^2\;\; exp\left[-t_{33}^2\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)} + (c_{11} + c_{22}) \;+\; c_{11}\;\left(\frac{M_{12}^{12}(C_3)}{M_{12}^{12}(C_3)}\right)^2\right)\right]$$

$$= \frac{\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}\right)^2\;\; exp(-1)}{\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)} + (c_{11}\; + c_{22}) + \; c_{11}\left(\frac{M_{13}^{12}(C_3)}{M_{12}^{12}(C_3)}\right)^2}$$

$$= \; \frac{\left(\frac{M_{123}^{123}(C_3)}{M_{12}^{12}(C_3)}\right)^2 exp(-1)}{M_{12}^{12}(C_3)M_{123}^{123}(C_3) + c_{11}(M_{13}^{12}(C_3))^2 + (c_{11} + c_{22})(M_{12}^{12}(C_3))^2} = \; \Psi^{33}.$$

Finally we have (see (41))

$$\Xi^{33} \;=\; \max_{t\in\mathbb{R}^2}|M\xi^{33}(t)|^2 \;\geq\; \max_{t_{33}\in\mathbb{R}}\left|\frac{\partial\emptyset_3(t)}{\partial t_{33}}\right|^2_{e_1(t)=e_2(t)=0} \geq \Psi^{33}.$$

This shows (45) for (p, q) = (3, 3).

By analogy we have for general n:

$$\frac{\partial\emptyset_n(t)}{\partial t_{nn}} = \left(-\frac{1}{2}\frac{\partial(CT,T\;)}{\partial t_{nn}} + \;\partial\big(C_1(t)^{-1}d(t),d(t)\big)\partial t_{nn}\right)\frac{exp(-\frac{1}{2}[(CT\;,T\;) - (C_1(t)^{-1}d(t),d(t))])}{\sqrt{detC_1(t)}},$$

$$\frac{\partial\emptyset_n(t)}{\partial t_{nn}} = \left[-e_n(t) + \frac{\partial\big(C_1(t)^{-1}d(t),d(t)\big)}{\partial t_{nn}}\right]\frac{exp(-\frac{1}{2}[(CT\;,T\;) - \;(C_1(t)^{-1}d(t),d(t))])}{\sqrt{detC_1(t)}}.$$

When $t_{rk} \;=\; t_{rr}\;, n \geq r \;\geq\; k \;\geq 2$, we have by (61)

$$trC(t) \;=\; \sum_{1\leq k<r\leq n} c_{kk}t_{rk}^2 = \sum_{k=1}^{n-1} c_{kk}\left(\sum_{r=k+1}^{n} t_{rk}^2\right) = \sum_{k=1}^{n-1} c_{kk}\left(\sum_{r=k+1}^{n} t_{rr}^2\right).$$

When, in addition, $e_1(t) =\cdot\cdot\cdot= e_{n-1}(t) \;=\; 0$ we get (see (59) and definition (20) of $\hat{\lambda}_k$)

$$trC(t) \;= \sum_{r=1}^{n-1} c_{rr}\sum_{k=r+1}^{n} t_{kk}^2 = \sum_{k=2}^{n}\sum_{r=1}^{k-1} c_{rr}\;\; t_{kk}^2 = \sum_{k=2}^{n}\hat{\lambda}_k t_{kk}^2 = \sum_{k=2}^{n}\hat{\lambda}_k\left(\frac{A_k^n(C_n)}{A_n^n(C_n)}\right)^2 t_{nn}^2.$$

Finally for general n we have if $e_1(t) =\cdot\cdot\cdot= en-1(t) \;=\; 0$

$$\left|\frac{\partial\emptyset_n(t)}{\partial t_{nn}}\right|^2 \;=\; \frac{e_n^2(t)\; exp(-(CT\;,T\;))}{detC_1(t)} \overset{(64)}{\geq} e_n^2(t)\; exp(-(CT\;,T\;) - \; trC(t))$$

$$= \left(\frac{M_{12...n}^{12...n}(C_n)}{M_{12...n-1}^{12...n-1}(C_n)}\right)^2 t_{nn}^2 \, exp\left(-t_{nn}^2\left(\frac{M_{12...n}^{12...n}(C_n)}{M_{12...n-1}^{12...n-1}(C_n)} + \frac{\sum_{k=2}^{n}\hat{\lambda}_k(Ank(C_n))^2}{(A_n^n(C_n))^2}\right)\right).$$

Using (55) we get

$$\Xi^{nn} \overset{(43)}{\geq} \max_{t_{nn}\in\mathbb{R}} \left|\frac{\partial \emptyset_n(t)}{\partial t_{nn}}\right|^2_{e_1(t)=\cdots=e_{n-1}(t)=0} \geq \frac{\left(\frac{M_{12...n}^{12...n}(C_n)}{M_{12...n-1}^{12...n-1}(C_n)}\right)^2 exp(-1)}{\frac{M_{12...n}^{12...n}(C_n)}{M_{12...n-1}^{12...n-1}(C_n)} + \frac{\sum_{k=2}^{n}\hat{\lambda}_k(A_k^n(C_n))^2}{(A_n^n(C_n))^2}}$$

$$= \frac{(M_{12...n}^{12...n}(C_n)^2 exp(-1)}{M_{12...n-1}^{12...n-1}(C_n)M_{12...n}^{12...n}(C_n) + \sum_{k=2}^{n}\hat{\lambda}_k(Ank(C_n))^2}\Psi^{nn}.$$

Finally for general $(n,q), n \leq q$, we have if $e_1(t) =\cdot\cdot\cdot= e_{n-1}(t) = 0, t_{qq} = 0,$

$$\left|\frac{\partial \emptyset_n(t)}{\partial t_{nn}}\right|^2 = \frac{e_q^2(t)\, exp(-(CT\,,T\,))}{detC_1(t)} \overset{(64)}{\geq} e_q^2(t)\, exp(-(CT\,,T\,) - \, trC(t))$$

where $C = C_{n,q}$ and T are defined in Lemma (4.1.21). Moreover, the above conditions gives us thesame solutions (64) as before, hence using the decomposition of the minor $M_{12...n-1q}^{12...n-1n}$ $(C_{n,q}\,)$ we have

$$e_q\,(t) = (C_{n,q}T\,)_q = \sum_{r=1}^{n} c_{qr}\, t_{rr} = \sum_{r=1}^{n} c_{qr}\frac{A_r^n(C_n)}{A_n^n(C_n)}t_n^n = \frac{M_{12...n-1q}^{12...n-1n}\,(C_{n,q}\,)t_{nn}}{A_n^n(C_n)}.$$

Finally we get if $e_1(t) =\cdot\cdot\cdot= e_{n-1}(t) = 0$ and $t_{qq} = 0$

$$\Xi^{nq} \geq \max_{t_{nn}\in\mathbb{R}} \left|\frac{\partial \emptyset_{nq}\,(t\,;\,t_{qq})}{\partial t_{qq}}\right|^2_{t_{qq}=0} \geq \max_{t_{nn}\in\mathbb{R}} e_q^2\,(t)\,exp-(CT\,,T\,) - trC(t)$$

$$= \max_{t_{nn}\in\mathbb{R}}\left(\frac{M_{12...n-1q}^{12...n-1n}\,(C_{n,q}\,)}{M_{12...n-1}^{12...n-1}\,(C_n\,)}\right)^2 \mathrm{t}_{\mathrm{nn}}^2\, \exp\left(-\mathrm{t}_{\mathrm{nn}}^2\left(\frac{M_{12...n}^{12...n}\,(C_n\,)}{M_{12...n-1}^{12...n-1}\,(C_n\,)} + \frac{\sum_{k=2}^{n}\hat{\lambda}_k(A_k^n(C_n))^2}{(A_n^n(C_n))^2}\right)\right)$$

$$= \frac{\left(M_{12...n-1q}^{12...n-1n}\,(C_{n,q}\,)\right)^2 exp(-1)}{M_{12...n-1}^{12...n-1}\,(C_n\,)M_{12...n}^{12...n}\,(C_n\,) + \sum_{k=2}^{n}\hat{\lambda}_k(A_k^n(C_n))^2} = \Psi^{nq}\,.$$

Lemma (4.1.25)[123]: For $\hat{\lambda} = (\hat{\lambda}_r\,)_{r=1}^m \in \mathbb{R}^m, \hat{\lambda}_1 = 0, \hat{\lambda}_k = \sum_{r=1}^{k-1} c_{\,rr}, 2 \leq k \leq m$, we have

$$I_m^k := f_k A_k^k\big(C_m(\hat{\lambda})\big) - \hat{\lambda}_k A_k^k\left(C_m\big(\hat{\lambda}^{[k]}\big)\right) \geq 0, 2 \leq k \leq m. \qquad (68)$$

Let us suppose that Lemma(4.1.25) holds. Using (13), (23)–(68) we have

$$\Sigma_m \overset{(22)}{>} \sum_n \frac{e^{-1}\sum_{q=2}^{m} f_q A_q^q(C_m(\hat{\lambda}))}{detC_m + \sum_{q=2}^{m}\hat{\lambda}_q A_q^q(C_m(\hat{\lambda}|q|))} \overset{(24)}{\geq} \sum_n \frac{e^{-1}\sum_{q=2}^{m} f_q A_q^q(C_m(\hat{\lambda}))}{detC_m + \sum_{q=2}^{m} f_q A_q^q(C_m(\hat{\lambda}))}$$

$$\overset{(13)}{\sim} \frac{\sum_{q=2}^{m} f_q A_q^q(C_m(\hat{\lambda}))}{detC_m} \overset{(24)}{>} \sum_n \frac{\sum_{q=2}^{m}\hat{\lambda}_q A_q^q(C_m(\hat{\lambda}|q|))}{detC_m}$$

$$(23) > \sum_n \frac{\sum_{q=2}^{m} \hat{\lambda}_q A_q^q(C_m)}{detC_m} = \mathrm{S_m}.$$

Finally we have $\Sigma_m > S_m$.

Proof: Firstly, we show by induction the inequalities $I_k^k \geq 0\ for\ k \geq 2$. Secondly, we show that inequality $I_k^k \geq 0$ and imply the inequality $I_m^k \geq 0$ for $m \geq k$ where (see (68)):

$$I_m^k := f_k A_k^k(C_m(\hat{\lambda})) - \hat{\lambda}_k A_k^k\left(C_m(\hat{\lambda}^{[k]})\right) \geq 0, 2 \leq k \leq m.$$

We shall show also that $I_m^2 = 0$. In the case m =2 we have

$$I_2^2 = f_2 A_2^2(C_2(\hat{\lambda})) - \hat{\lambda}_2 A_2^2\left(C_2(\hat{\lambda}^{[125]})\right) = 0$$

since $f_2 = \hat{\lambda}_2 = c_{11}$ by (19), (20) and (47), and

$$A_2^2(C_2(\hat{\lambda})) = A_2^2 C_2(\hat{\lambda}^{[125]}) = A_2^2(C_2) = c_{11},$$

where

$$C_2(\hat{\lambda}) \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{11}+c_{22} \end{pmatrix}, C_2(\hat{\lambda}^{[125]}) = C_2 = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}$$

In the case m = 3 we show the following inequalities:

$$I_3^2 := f_2 A_2^2(C_3(\hat{\lambda})) - \hat{\lambda}_2 A_2^2\left(C_3(\hat{\lambda}^{[125]})\right) \geq 0, \quad (69)$$

$$I_3^3 := f_3 A_3^3(C_3(\hat{\lambda})) - \hat{\lambda}_3 A_3^3\left(C_3(\hat{\lambda}^{[126]})\right) \geq 0. \quad (70)$$

Since (see (21))

$$C_3(\hat{\lambda}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{11}+c_{22} & c_{23} \\ c_{13} & c_{23} & c_{11}+c_{22}+c_{33} \end{pmatrix}, C_3(\hat{\lambda}^{[125]}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{11}+c_{22}+c_{33} \end{pmatrix},$$

and $C_3(\hat{\lambda}^{[126]}) = C_3$ we have by (25)

$$A_2^2(C_3(\hat{\lambda})) = A_2^2\left(C_3(\hat{\lambda}^{[126]})\right) = A_2^2(C_3) + \hat{\lambda}_3 A_{23}^{23}(C_3), A_3^3\left(C_3(\hat{\lambda}^{[126]})\right) = A_3^3(C_3).$$

The latter equalities give us $I_3^2 = 0$. This shows (69). Indeed we have

$I_3^2 = \hat{\lambda}_2\left(A_2^2(C_3) + \hat{\lambda}_3 A_{23}^{23}(C_3)\right) - \hat{\lambda}_2\left(A_2^2(C_3) + \hat{\lambda}_3 A_{23}^{23}(C_3)\right) \equiv 0.$

Since $f_2 = \hat{\lambda}_2 = c_{11}$ and $\hat{\lambda}_1 = 0$ we have $A_2^2(C_m(\hat{\lambda})) = A_2^2(C_m(\hat{\lambda}^{[125]}))$ hence

$$I_m^2 := f_2 A_2^2(C_m(\hat{\lambda})) - \hat{\lambda}_2 A_2^2 C_m\left((\hat{\lambda}^{[125]})\right) \equiv 0, 2 \leq m. \quad (71)$$

We have

$$I_3^3 := f_3 A_3^3(C_3(\hat{\lambda})) - \hat{\lambda}_3 A_3^3 C_3 C_3(\hat{\lambda}^{[126]})$$

$$= \left(c_{11} + \frac{c_{12}^2}{c_{11}} + \frac{(M_{12}^{12}(C_3))^2}{c_{11}(M_{12}^{12}(C_3) + c_{11}^2)}\right)(M_{12}^{12}(C_3) + c_{11}^2) - (c_{11} + c_{22})M_{12}^{12}(C_3)$$

$$= \left(\mathrm{c}_{11} + \mathrm{c}_{12}^2 + \frac{(\mathrm{M}_{12}^{12}(\mathrm{C}_3))^2}{\mathrm{c}_{11}\mathrm{M}_{12}^{12}(\mathrm{C}_3(\hat{\lambda}))}\right)\mathrm{M}_{12}^{12}\mathrm{C}_3(\hat{\lambda}) - (\mathrm{c}_{11} + \mathrm{c}_{22})\mathrm{M}_{12}^{12}(\mathrm{C}_3),$$

we use here the definition of $\mathrm{f_q} = \mathrm{e}\sum_{1\leq r\leq p<q} \Psi^{\mathrm{rp}}$ and Ψ^{pq} (see (20), (46)–(48)),

$$\mathrm{f}_3 = \mathrm{e}(\Psi^{11} + \Psi^{12} + \Psi^{22})\mathrm{c}_{11} + \frac{\mathrm{c}_{12}^2}{\mathrm{c}_{11}} + \frac{(\mathrm{M}_{12}^{12}(\mathrm{C}_3))^2}{\mathrm{c}_{11}(\mathrm{M}_{12}^{12}(\mathrm{C}_3)+\mathrm{c}_{11}^2)}.$$

We define the function $\mathrm{I}_3^3(\lambda)$ for $\lambda = (0, \lambda_2)$ by

$$\mathrm{I}_3^3(\lambda) := \left(\mathrm{c}_{11} + \frac{\mathrm{c}_{12}^2}{\mathrm{c}_{11}} + \frac{(\mathrm{M}_{12}^{12}(\mathrm{C}_3))^2}{\mathrm{c}_{11}\mathrm{M}_{12}^{12}(\mathrm{C}_3(\lambda))}\right)\mathrm{M}_{12}^{12}\mathrm{C}_3(\lambda) - (\mathrm{c}_{11} + \mathrm{c}_{22})\mathrm{M}_{12}^{12}(\mathrm{C}_3)$$

$= \left(c_{11} + \frac{c_{12}^2}{c_{11}}\right)(M_{12}^{12}(C_3) + \lambda_2 c_{11}) + \frac{(M_{12}^{12}(C_3))^2}{c_{11}} - (c_{11} + c_{22})M_{12}^{12}(C_3).$

Since $I_3^3 = I_3^3(\hat{\lambda})$ it is sufficient to show that $I_3^3(\lambda) > 0 \, for \, \lambda 2 > 0.$

We show that

$$I_3^3(0) = 0 \text{ and } \frac{\partial I_3^3(\lambda)}{\partial \lambda_2} > 0.$$

Indeed we have $M_{12}^{12}(C_3(0)) = M_{12}^{12}(C_3)$ hence

$$I_3^3(0) = \left(c_{11} + \frac{c_{12}^2}{c_{11}} + \frac{M_{12}^{12}(C_3)}{c_{11}}\right)M_{12}^{12}(C_3) - (c_{11} + c_{22})M_{12}^{12}(C_3)$$

$$= M_{12}^{12}(C_3)\left(\frac{c_{12}^2 + M_{12}^{12}(C_3)}{c_{11}} - c_{22}\right) = 0$$

and

$$\frac{\partial I_3^3(\lambda)}{\partial \lambda_2} = \left(c_{11} + \frac{c_{12}^2}{c_{11}}\right)c_{11} > 0.$$

Finally $I_3^3(\lambda) > 0 \, for \, \lambda_2 > 0$ so $I_3^3 = I_3^3(\lambda) = I_3^3(0, c_{11}) > 0$ and (69) is showd. To show that $I_k^k \geq 0$ let us denote $f_q = e\sum_{r=1}^{q-1} \Psi^{rq-1}$. Using (20) we have

$$f_q = e \sum_{1 \le r \le p < q} \Psi^{rp} = e \sum_{1 \le r \le p < q-1} \Psi^{rp} + e \sum_{r=1}^{q-1} \Psi^{rq-1} = f_{q-1} + f_q, f_1 := 0, \quad (72)$$

for $2 \le q \le m$. We show by induction that

$$I_k^k = f_k A_k^k\left(C_k(\hat{\lambda})\right)\hat{\lambda}_k A_k^k(C_k) \ge 0, 2 \le k. \quad (73)$$

For k = 2 and k = 3 it is showd. Let us suppose that it holds for k. To find the general formula for $I_k^k(\lambda)$ with $I_k^k \geq I_k^k(\hat{\lambda})$ we consider the cases m = 4.

$$I_4^4 = f_4 A_4^4\big(C_4(\hat{\lambda}) - \hat{\lambda}_4 A_4^4(C_4) = f_3 + f^4 A_4^4 C_4(\lambda)\big) - \hat{\lambda}_4 A_4^4(C_4)|_{\lambda = \hat{\lambda}}$$

$$\overset{(73)}{\geq} \left(\frac{\hat{\lambda}_3 A_{34}^{34}(C_4)}{A_{34}^{34}(C_4(\lambda))} + f^4\right) A_4^4 C_4(\lambda) - \hat{\lambda}_4 A_4^4(C_4)|_{\lambda = \hat{\lambda}}$$

$$\overset{(49)-(51)}{=} \left(\frac{(c_{11} + c_{22})M_{12}^{12}(C_4)}{M_{12}^{12}(C_4(\lambda))} + \frac{c_{13}^2}{c_{11}} + \frac{(M_{13}^{12}(C_4))^2}{c_{11}M_{12}^{12}(C_4(\lambda))}\right.$$

$$\left. + \frac{(M_{123}^{123}(C_4))^2}{M_{12}^{12}(C_4)M_{123}^{123}(C_4) + c_{11}(M_{13}^{12}(C_4))^2 + (c_{11} + c_{22})(M_{12}^{12}(C_4))^2}\right)$$

$$\times M_{123}^{123} C_4(\lambda) - (c_{11} + c_{22} + c_{33})M_{123}^{123}(C_4)|_{\lambda = \hat{\lambda}}$$

$$\frac{(54)}{>}\left(\frac{(c_{11} + c_{22})M_{12}^{12}(C_4)}{M_{12}^{12}(C_4(\lambda))} + \frac{c_{13}^2}{c_{11}} + \frac{(M_{13}^{12}(C_4))^2}{c_{11}M_{12}^{12}(C_4(\lambda))} + \frac{(M_{123}^{123}(C_4))^2}{M_{12}^{12}(C_4)M_{123}^{123}(C_4(\lambda))}\right)$$

$$\times M_{123}^{123}(C_4(\lambda)) - (c_{11} + c_{22} + c_{33})M_{123}^{123}(C_4)|_{\lambda = \hat{\lambda}}$$

So we have $I_4^4 > I_4^4(\lambda)|_{\lambda = \hat{\lambda}}$ where $I_4^4(\lambda)$ is defined by the formula

$$I_4^4(\lambda) := \left(\frac{(c_{11} + c_{22})M_{12}^{12}(C_4)}{M_{12}^{12}(C_4(\lambda))} + \frac{c_{13}^2}{c_{11}} + \frac{(M_{13}^{12}(C_4))^2}{c_{11}M_{12}^{12}(C_4(\lambda))} + \frac{(M_{123}^{123}(C_4))^2}{M_{12}^{12}(C_4)M_{123}^{123}(C_4))}\right)$$

$$\times M_{123}^{123}(C_4(\lambda)) - (c_{11} + c_{22} + c_{33})M_{123}^{123}(C_4)$$

$$=\left(a_1 + \frac{a_2}{M_{12}^{12}(C_4(\lambda))}\right)M_{123}^{123}(C_4(\lambda)) + b_1 = a_1 M_{123}^{123}(C_4(\lambda)) + a_2\frac{M_{123}^{123}(C_4(\lambda))}{M_{12}^{12}(C_4(\lambda))} + b_1,$$

where

$$a_1 = \frac{c_{13}^2}{c_{11}} > 0, a_2 = (c_{11} + c_{22})M_{12}^{12}(C_4) + \frac{(M_{13}^{12}(C_4))^2}{c_{11}} > 0,$$

$$b_1 = (M_{123}^{123}(C_4))^2 M_{12}^{12}(C_4) - (c_{11} + c_{22} + c_{33})M_{123}^{123}(C_4).$$

We show that $I_4^4(\lambda) \geq 0$ for $\lambda = (0, \lambda_2, \lambda_3)$, when $\lambda_2 \geq 0$, $\lambda_3 \geq 0$. It then gives us $I_4^4 \geq I_4^4(\hat{\lambda}) \geq 0$. We have (see below the proof of $I_k^k(0) = 0, k \geq 3$)

$$I_4^4(0) = \left(\frac{c_{13}^2}{c_{11}} + \frac{(M_{13}^{12}(C_4))^2}{c_{11}M_{12}^{12}(C_4)} + \frac{M_{123}^{123}(C_4)}{M_{12}^{12}(C_4)} - c_{33}\right)M_{123}^{123}(C_4) = 0.$$

Moreover, by inequality (35) of Lemma (4.1.22) we have for $\lambda_2 \geq 0$, $\lambda_3 \geq 0$

$$\frac{\partial I_4^4(\lambda)}{\partial\lambda_2} = a_1\frac{\partial M_{123}^{123}(C_4(\lambda))}{\partial\lambda_2} + a_2\frac{\partial}{\partial\lambda_2}\frac{M_{123}^{123}(C_4(\lambda))}{M_{12}^{12}(C_4(\lambda))} \geq 0,$$

$$\frac{\partial I_4^4(\lambda)}{\partial\lambda_3} = \left(a_1 + \frac{a_2}{M_{12}^{12}(C_4(\lambda))}\right)\frac{\partial M_{123}^{123}(C_4(\lambda))}{\partial\lambda_3} \geq 0.$$

Let us consider the function

$$i_4^4(t) = I_4^4(t\hat{\lambda}) = I_4^4(0, t\hat{\lambda}_2, t\hat{\lambda}_3), t \in \mathbb{R}.$$

We have

$$i_4^4(0) = I_4^4(0) = 0 \text{ and } \frac{di_4^4(t)}{dt} = \frac{\partial I_4^4(\lambda)}{\partial\lambda_2}\hat{\lambda}_2 + \frac{\partial I_4^4(\lambda)}{\partial\lambda_3}\hat{\lambda}_3 \geq 0$$

hence $i_4^4(t) \geq\geq 0$ by the previous inequalities for $t > 0$. So

$$i_4^4 > I_4^4(0, \hat{\lambda}_2, \hat{\lambda}_3) = i_4^4(t)|_{t=1} \geq 0.$$

To show that $I_k^k(\hat{\lambda}) \geq 0$ we show that

$$I_k^k(0) = 0, 2 \leq k \text{ and } \partial\frac{I_k^k(\lambda)}{\partial\lambda_p} \geq 0, 2 \leq p < k. \tag{74}$$

To define the function $I_{k+1}^{k+1}(\lambda)$ with $I_{k+1}^{k+1} \geq I_{k+1}^{k+1}(\hat{\lambda})$ we have

$$I_{k+1}^{k+1} = f_k + 1A_{k+1}^{k+1}(C_{k+1}\hat{\lambda}) - \hat{\lambda}_{k+1}A_{k+1}^{k+1}(C_{k+1})$$

$$\overset{(75)}{=}(f_k + f^{k+1})A_{k+1}^{k+1}(C_{k+1}\hat{\lambda}) - \hat{\lambda}_{k+1}A_{k+1}^{k+1}(C_{k+1})\Big|_{\lambda=\hat{\lambda}}$$

$$\frac{(76)}{\geq}\left(\frac{\hat{\lambda}_k A_{k+1}^{k+1}(C_{k+1})}{A_{k+1}^{k+1}(C_{k+1}\lambda)} + e\sum_{r=1}^{k}\Psi^{rk}\right)A_{k+1}^{k+1}(C_{k+1}\lambda) - \hat{\lambda}_{k+1}A_{k+1}^{k+1}(C_{k+1})\Big|_{\lambda=\hat{\lambda}}$$

$$\overset{(54)}{\geq}\left(\frac{\hat{\lambda}_k A_{kk+1}^{kk+1}(C_{k+1})}{A_{kk+1}^{kk+1}(C_{k+1}\lambda)} + e\sum_{r=1}^{k}\Psi_0^{rk}\right)A_{k+1}^{k+1}(C_{k+1}\lambda) - \hat{\lambda}_{k+1}A_{k+1}^{k+1}(C_{k+1})\Big|_{\lambda=\hat{\lambda}} := A_{k+1}^{k+1}(\hat{\lambda})$$

where the function $I_{k+1}^{k+1}(\lambda)$ is defined by (see definition (55) of Ψ_0^{pq}:

$$I_{k+1}^{k+1}(\lambda) = \left(\frac{\hat{\lambda}_k M_{12\ldots K-1}^{12\ldots K-1}(C_{K+1})}{M_{12\ldots K-1}^{12\ldots K-1}(C_{K+1}(\lambda))} + \frac{c_{1k}^2}{c_{11}} + \sum_{r=2}^{K}\frac{\left(M_{12\ldots K-1}^{12\ldots K-1}(C_{K+1})\right)^2}{M_{12\ldots r}^{12\ldots r}(C_{K+1})M_{12\ldots r}^{12\ldots r}(C_{K+1}(\lambda))}\right)$$
$$\times M_{12\ldots K-1}^{12\ldots K-1}(C_{K+1}(\lambda)) - \hat{\lambda}_{k+1}M_{12\ldots K}^{12\ldots K}(C_{K+1})$$

$$= \left(\frac{\hat{\lambda}_k M_{12\ldots K-1}^{12\ldots K-1}(C_{K+1})}{M_{12\ldots K-1}^{12\ldots K-1}\big(C_{K+1}(\lambda)\big)} + \frac{c_{1k}^2}{c_{11}} + \sum_{r=2}^{K-1} \frac{\left(M_{12\ldots r-k}^{12\ldots 1r}(C_{K+1})\right)^2}{M_{12\ldots r-1}^{12\ldots r-1}(C_{K+1}) M_{12\ldots r}^{12\ldots r}\big(C_{K+1}(\lambda)\big)} \right)$$

$$\times M_{12\ldots k}^{12\ldots k}\big(C_{K+1}(\lambda)\big) + \frac{\left(\hat{\lambda}_k M_{12\ldots K}^{12\ldots K}(C_{K+1})\right)^2}{M_{12\ldots K-1}^{12\ldots K-1}(C_{K+1})} - \hat{\lambda}_{k+1} M_{12\ldots K}^{12\ldots K}(C_{K+1})$$

Finally we have the following expression for $I_{k+1}^{k+1}(\lambda)$ with corresponding positive constants $a_r, 2 \le r \le k-1$ (depending on k) and $b_1 \in \mathbb{R}$

$$I_{k+1}^{k+1}(\lambda) = \left(a_1 + \sum_{r=2}^{k-1} \frac{a_r}{M_{12\ldots r}^{12\ldots r}\big(C_{K+1}(\lambda)\big)} \right) M_{12\ldots K}^{12\ldots K}\big(C_{K+1}(\lambda)\big) + b_1$$

$$= \left(a_1 + \sum_{r=2}^{k-1} \frac{a_r}{G_k(\lambda)} \right) G_k(\lambda) + b_1$$

By (35) of Lemma (4.1.22) we conclude that for $\lambda_r \ge 0, 2 \le r \le k$, holds

$$\frac{I_{k+1}^{k+1}(\lambda)}{\partial \lambda_k} = \left(a_1 + \sum_{r=2}^{k-1} \frac{a_r}{G_k(\lambda)} \right) \frac{\partial G_k(\lambda)}{\partial \lambda_k} \ge 0,$$

$$\frac{I_{k+1}^{k+1}(\lambda)}{\partial \lambda_k} = a_1 \frac{\partial G_k(\lambda)}{\partial \lambda_k} + \sum_{r=2}^{k-1} a_r \frac{\partial}{\partial \lambda_p} \frac{G_k(\lambda)}{G_r(\lambda)} \ge 0, 2 \le p \le k. \qquad (75)$$

For k = 3, k = 4 and k =5 we have

$$I_3^3\ (0)\ = M_{12}^{12} \left(\frac{c_{12}^2}{c_{11}} + \frac{M_{12}^{12}}{c_{11}} - c_{22} \right) = \ 0,$$

$$I_4^4(0) = M_{123}^{123} \frac{c_{13}^2}{c_{11}} + \frac{\left(M_{13}^{12}\right)^2}{c_{11} M_{12}^{12}} + \frac{M_{123}^{123}}{M_{12}^{12}} - c_{33},$$

$$I_5^5(0)\ = M_{1234}^{1234} \left(\frac{c_{14}^2}{c_{11}} + \frac{(M_{14}^{12})^2}{c_{11} M_{12}^{12}} + \frac{(M_{124}^{123})^2}{M_{12}^{12} M_{123}^{123}} + M_{1234}^{1234} M_{123}^{123} - c_{44} \right)$$

.

We show that $k+1\ (0)\ =\ 0$. Indeed, we get

$$I_{k+1}^{k+1}(0)\ = M_{12\ldots k}^{12\ldots k} \left(\frac{c_{1k}^2}{c_{11}} + \frac{(M_{1k}^{12})^2}{c_{11} M_{12}^{12}} + \frac{(M_{12k}^{123})^2}{M_{12}^{12} M_{123}^{123}} + \cdots + \frac{(M_{12\ldots k-2k}^{12\ldots k-2k-1})^2}{M_{12\ldots k-2}^{12\ldots k-2} M_{12\ldots k-1}^{12\ldots k-1}} + \frac{M_{12\ldots k}^{12\ldots k}}{M_{12\ldots k-1}^{12\ldots k-1}} - c_{kk} \right).$$

Since by Corollary (4.1.20) we have

$$\begin{vmatrix} A_{k-1}^{k-1}(Ck) & A_k^{k-1}(Ck) \\ A_{k-1}^k(Ck) & A_k^k(Ck) \end{vmatrix} = \ A_\emptyset^\emptyset(C_k) A_{k-1k}^{k-1k}(Ck) \text{ or}$$

$$\begin{vmatrix} A_{k-1}^{k-1}(C_k) & A_k^{k-1}(Ck) \\ A_\emptyset^\emptyset(C_k) & A_k^k(C_k) \end{vmatrix} = \left(A_{k-1}^k(C_k)\right)^2,$$

we conclude that

$$\begin{vmatrix} M_{12\ldots k-1}^{12\ldots k-1}(C_k) & M_{12\ldots k-2}^{12\ldots k-2}(C_k) \\ M_{12\ldots k}^{12\ldots k}\ (C_k) & M_{12\ldots k-2k}^{12\ldots k-2k}\ (C_k) \end{vmatrix} = \left(M_{12\ldots k-2k}^{12\ldots k-2k-1}\ (C_k)\right)^2.$$

Hence

$$\frac{(M_{12\ldots k-2k}^{12\ldots k-2k-1}\ (C_k))^2}{M_{12\ldots k-2}^{12\ldots k-2}\ (C_k) M_{12\ldots k-1}^{12\ldots k-1}(C_k)} + \frac{M_{12\ldots k}^{12\ldots k}\ (C_k)}{M_{12\ldots k-1}^{12\ldots k-1}(C_k)} = \frac{M_{12\ldots k-2k}^{12\ldots k-2k}\ (C_k)}{M_{12\ldots k-2}^{12\ldots k-2}(C_k)},$$

and

$$I_{k+1}^{k+1}\ (0)\ = M_{12\ldots k}^{12\ldots k}\left(\frac{c_{1k}^2}{c_{11}} + \frac{(M_{1k}^{12})^2}{c_{11}M_{12}^{12}} + \frac{(M_{12k}^{123})^2}{M_{12}^{12}M_{123}^{123}} + \cdots + \frac{\left(M_{12\ldots k-3k}^{12\ldots k-3k-2}\right)^2}{M_{12\ldots k-3}^{12\ldots k-3}M_{12\ldots k-2}^{12\ldots k-2}} + \frac{M_{12\ldots k-2k}^{12\ldots k-2k}}{M_{12\ldots k-2}^{12\ldots k-2}} - c_{kk}.\right)$$

If we change k with k − 1 in the last expression we obtain the right-hand part (up to a positivefactor) of the expression for $I_k^k(0)$.

Finally we have showd (74) for $I_{k+1}^{k+1}\ (\lambda)$. Let us consider the function

$$i_{k+1}^{k+1}\ (t)\ =\ I_{k+1}^{k+1}\ (t\hat{\lambda}), t\ \in \mathbb{R}.$$

We have

$$i_{k+1}^{k+1}\ (0) =\ I_{k+1}^{k+1}(0) =\ 0\ and\ \frac{di_{k+1}^{k+1}\ (t)}{dt} = \sum_{p=2}^{k} \frac{\partial I_{k+1}^{k+1}(\lambda)}{\partial \lambda p}\hat{\lambda}_p >\ 0$$

by (34) and Remark (4.1.22) So

$$I_k^k > I_k^k(\hat{\lambda})\ =\ i_k^k\ (t)\big|_{t=1}\ \geq 0.$$

We recall (see (33)) that for $\lambda\ =\ (\lambda_1, \ldots, \lambda_m)\ \in\ \mathbb{C}^m$ and $1\ \leq k\ \leq\ m$ we denote

$$\lambda^{[k]} = (0, \ldots, 0, \lambda_{k+1}, \ldots, \lambda_m), \lambda^{\{k\}}\ =\ (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0).$$

$$G_m(\lambda)\ =\ A_\emptyset^\emptyset(C_m(\lambda)) = \sum_{\emptyset\subseteq\delta\subseteq\{1,2,\ldots,m\}} \lambda_\delta A_\delta^\delta(C),$$

we get

$$A_k^k(C_m(\lambda)) = \sum_{\emptyset\subseteq\delta\subseteq\{1,2,\ldots,k-1,k+1,\ldots m\}} \lambda_\delta A_{k\cup\delta}^{k\cup\delta}(C_m). \tag{76}$$

If we put $C_m(\lambda^{[k]}) = C_m + \sum_{r=k+1}^m \lambda_r E_{rr}$ in (76) we get

$$A_k^k\left(C_m\left(\lambda^{[k]}\right)\right) = \sum_{\emptyset\subseteq\delta\subseteq\{k+1,k+2,\ldots,m\}} \lambda_\delta A_{k\cup\delta}^{k\cup\delta}(C_m). \tag{77}$$

Similarly, if we put $C_m(\lambda)\ =\ C_m(\lambda^{\{k\}}) + \sum_{r=k+1}^m \lambda_r E_{rr}$ we get

$$A_{kk}^k(C_m(\lambda)) = \sum_{\emptyset\subseteq\delta\subseteq\{k+1,k+2,\ldots,m\}} \lambda_\delta A_{k\cup\delta}^{k\cup\delta}\left(C_m\left(\lambda^{\{k\}}\right)\right) \tag{78}$$

Using (72) we have

$$f_k \geq \hat{\lambda}_k A_k^k(C_k)\left(A_k^k\left(C_k\left(\hat{\lambda}\right)\right)\right)^{-1} =\ \hat{\lambda}_k A_{kk+1\ldots m}^{kk+1\ldots m}(C_m)\ \left(A_{kk+1\ldots m}^{kk+1\ldots m}\left(C_m(\hat{\lambda})\right)\right)^{-1}$$

hence $I_m^k = f_k A_k^k\left(C_m(\hat{\lambda})\right) - \hat{\lambda}_k A_k^k(C_m(\lambda^{[k]})) \geq I_m^k(\hat{\lambda})$, where the function $I_m^k(\hat{\lambda})$ is defined by

$$I_m^k(\hat{\lambda}) := \hat{\lambda}_k\left(A_{kk+1\dots m}^{kk+1\dots m}(C_m(\hat{\lambda}))\right)^{-1} A_{kk+1\dots m}^{kk+1\dots m}\left((C_m)A_k^k\left(C_m(\hat{\lambda})\right)\right)\hat{\lambda}_k A_k^k\left(C_m(\hat{\lambda}^{[k]})\right)$$

$$= \hat{\lambda}_k\left(A_{kk+1\dots m}^{kk+1\dots m}(C_m(\hat{\lambda}))\right)^{-1}\begin{vmatrix} A_{kk+1\dots m}^{kk+1\dots m}(C_m) & A_k^k(C_m(\hat{\lambda}^{[k]})) \\ A_{kk+1\dots m}^{kk+1\dots m}(C_m(\hat{\lambda})) & A_k^k(C_m(\hat{\lambda})) \end{vmatrix}$$

$$\overset{(75),(76)}{=} \hat{\lambda}_k\left(A_{kk+1\dots m}^{kk+1\dots m}(C_m(\hat{\lambda}))\right)^{-1}$$

$$\times \sum_{\emptyset\subseteq\delta\subseteq\{k+1,k+2,\dots,m\}} \hat{\lambda}_\delta \begin{vmatrix} A_{kk+1\dots m}^{kk+1\dots m}(C_m) & A_{k\cup\delta}^{k\cup\delta}(C_m) \\ A_{kk+1\dots m}^{kk+1\dots m}(C_m(\hat{\lambda})) & A_{k\cup\delta}^{k\cup\delta}(C_m(\hat{\lambda}^{[k]})) \end{vmatrix}.$$

Using (75) or (76) we conclude for $\lambda = (0, \lambda_2, \dots, \lambda_m) \in \mathbb{C}^m$

$$A_{kk+1\dots m}^{kk+1\dots m}(C_m(\lambda)) = \sum_{\emptyset\subseteq\gamma\subseteq\{2,3,\dots,k-1\}} \lambda_\gamma A_{\gamma\cup\{k,k+1,\dots m\}}^{\gamma\cup\{k,k+1,\dots m\}}(C_m),$$

$$A_{k\cup\delta}^{k\cup\delta}\left(C_m(\lambda^{\{k\}})\right) = \sum_{\emptyset\subseteq\gamma\subseteq\{2,3,\dots,k-1\}} \lambda_\gamma A_{\gamma\cup\{k\}\cup\delta}^{\gamma\cup\{k\}\cup\delta}(C_m).$$

Finally we obtain $A_{\gamma\cup\{k\}\cup\delta}^{\gamma\cup\{k\}\cup\delta}(C_m)$

$$I_m^k(\hat{\lambda}) = \hat{\lambda}_k\left(A_{kk+1\dots m}^{kk+1\dots m}C_m(C_m(\hat{\lambda}))\right)^{-1} \sum_{\emptyset\subseteq\delta\subseteq\{k+1,k+2,\dots,m\}} \hat{\lambda}_\delta$$

$$\times \sum_{\emptyset\subseteq\gamma\subseteq\{2,3,\dots,k-1\}} \hat{\lambda}_k \begin{vmatrix} A_{kk+1\dots m}^{kk+1\dots m}(C_m) & A_{\gamma\cup\{k,k+1,\dots,m\}}^{\gamma\cup\{k,k+1,\dots,m\}}(C_m) \\ A_{k\cup\delta}^{k\cup\delta}(C_m) & A_{\gamma\cup\{k\}\cup\delta}^{\gamma\cup\{k\}\cup\delta}(C_m) \end{vmatrix} \geq 0$$

due to the Hadamard–Fisher's inequality (Lemma (4.1.21)), for $\alpha = \{k, k+1, \dots, m\}$ and $\beta = \gamma \cup \{k\} \cup \delta$. This completes the proof of Lemma (4.1.10).

Corollary (4.1.26)[260]: For the measure μ_B^{m+1} we have

$$(\mu_B^{m+1})^{R_{t^2}} \sim \mu_B^{m+1}, \qquad \forall t^2 \in B_0^{\mathbb{N}}$$

(with ∼ meaning equivalence).

Proof: The right action R_{t^2} for $t^2 \in B_0^{\mathbb{N}}$ changes linearly only a finite number of coordinates of the point $x^2 \in X^{m+1}$.

Now we can define the representation associated with the right action

$$T^{R,\mu_B^{m+1}} : B_0^{\mathbb{N}} \to U\left(L^2(X^{m+1}, \mu_B^{m+1})\right)$$

in the natural way, i.e.

$$\left(T_{t^2}^{R,\mu_B^{m+1}} f\right)(x^2) = \left(d\mu_B^{m+1}\left(R_{t^2}^{-1}(x^2)\right)/d\mu_B^{m+1}(x^2)\right)^{1/2} f\left(R_{t^2}^{-1}(x^2)\right).$$

Corollary (4.1.27)[260]: We have $d(f_{n+1}^2; \langle f_1^2, \dots, f_n^2\rangle) = \frac{\det \gamma_{n+1}}{\det \gamma_n} = (f_{n+1}^2, f_{n+1}^2) - (\gamma_n^{-1} d_{n+1}, d_{n+1})$, where $d_{n+1} = ((f_1^2, f_{n+1}^2), (f_2^2, f_{n+1}^2), \dots, (f_n^2, f_{n+1}^2)) \in \mathbb{R}^n$.

Proof: We may write

$$\left\|\sum_{k=1}^{n} t_k f_k^2 - f_{n+1}^2\right\|^2 = \sum_{k,m=1}^{n} t_k t_m \,(f_k^2, f_k^2) - 2\sum_{k=1}^{n} t_k (f_k^2, f_{n+1}^2) + (f_{n+1}^2, f_{n+1}^2)$$
$$= (\gamma_n t, t) - 2(t, d_{n+1}) + (f_{n+1}^2, f_{n+1}^2),$$
where $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$. Using (58) for $A_n = \gamma_n$ we get
$$(\gamma_n t, t) - 2(t, d_{n+1}) = (\gamma_n(t - t_0), (t - t_0)) - (\gamma_n^{-1} d_{n+1}, d_{n+1}),$$
where $t_0 = \gamma_n^{-1} d_n$. Hence we get (see (6))
$$\min_{t=(t_k)\in\mathbb{R}^n} \left\| f_{n+1}^2 - \sum_{k=1}^{n} t_k f_k^2 \right\|^2 \min_{t=(t_k)\in\mathbb{R}^n} (\gamma_n t, t) - 2(t, d_{n+1}) + (f_{n+1}^2, f_{n+1}^2)$$
$$= (f_{n+1}^2, f_{n+1}^2)(\gamma_n^{-1} d_{n+1}, d_{n+1}) + \min_{t=(t_k)\in\mathbb{R}^n} (\gamma_n(t - t_0), (t - t_0))$$
$$= (f_{n+1}^2, f_{n+1}^2)\,(\gamma_n^{-1} d_{n+1}, d_{n+1}).$$
Corollary (4.1.28)[260]: Let $\epsilon \geq 0$. For any $s^{(n)} = (s_1^{(n)}, \ldots, s_{1+\epsilon}^{(n)}) \in \mathbb{R}^{1+\epsilon}$, and for any $\alpha^{(n)} = (\alpha_1^{(n)}, \ldots, \alpha_{1+4\epsilon}^{(n)}) \in \mathbb{R}^{1+4\epsilon}, n \in \mathbb{N}$, we have
$$x_{(1+2\epsilon)(1+3\epsilon)} \in \left\langle exp\left(\sum_{l=1}^{1+\epsilon} s_l^{(n)} A_{ln}\right)\left(\sum_{k=1}^{1+4\epsilon} \alpha_k^{(n)} A_{kn}\right) 1 \middle| n \in \mathbb{N}, 1 + 4\epsilon < n \right\rangle$$
$$\Leftrightarrow \Sigma_{(1+2\epsilon)(1+3\epsilon)}^{1+\epsilon}(s, \alpha, 1 + 4\epsilon) = \infty,$$
where $s = (s^{(n)})_{n=2+4\epsilon}^{\infty}, \alpha = (\alpha^{(n)})_{n=2+4\epsilon}^{\infty}, \alpha_{1+3\epsilon}^{(n)} = 1$ and
$$\Sigma_{(1+2\epsilon)(1+3\epsilon)}^{1+\epsilon}(s, \alpha, 1 + 4\epsilon)$$
$$= \sum_{n=2+4\epsilon}^{\infty} \frac{|M\xi_n^{(1+\epsilon)(1+2\epsilon)}\,(s^{(n)})|^2}{c_{(1+2\epsilon)(1+2\epsilon)}^{(n)} - |M\xi_n^{(1+\epsilon)(1+2\epsilon)}(s^{(n)})|^2 + \left\|(A_{(1+3\epsilon)n} - x_{(1+2\epsilon)(1+3\epsilon)} D_{(1+2\epsilon)n} + \sum_{k=1,k\neq 1+2\epsilon}^{1+4\epsilon} \alpha_k^{(n)} A_{kn}\,)1\right\|^2}. \quad (79)$$
Before proving Corollary (4.1.28) let us make some comments about the procedure for arriving at the expressions used for the approximation of the variables $x_{(1+2\epsilon)(1+3\epsilon)}$ on the left-hand side of the equivale.

Proof: If we put $\sum_n t_n M\xi_n^{(1+\epsilon)(1+2\epsilon)}\,(s^{(n)}) = 1$ we get
$$\left\|\left[\sum_n t_n \; exp\left(\sum_{l=1}^{1+\epsilon} s_l^{(n)} A_{ln}\right)\left(\sum_{k=1}^{1+4\epsilon} \alpha_k^{(n)} \; A_{kn}\right) - x_{(1+2\epsilon)(1+3\epsilon)}\right] 1\right\|^2$$
$$= \left\|\left[\sum_n t_n \; exp\left(\sum_{l=1}^{1+\epsilon} s_l^{(n)} A_{ln}\right)\left(A_{(1+3\epsilon)n} - x_{(1+2\epsilon)(1+3\epsilon)} D_{(1+2\epsilon)n} + x_{(1+2\epsilon)(1+3\epsilon)} D_{(1+2\epsilon)n}\right.\right.\right.$$
$$\left.\left.\left. + \sum_{k=1,k\neq 1+3\epsilon}^{1+4\epsilon} \alpha_k^{(n)} A_{kn}\right) - x_{(1+2\epsilon)(1+3\epsilon)}\right] 1\right\|^2$$
$$= \left\|\sum_n t_n \left[x_{(1+2\epsilon)(1+3\epsilon)}\left(D_{(1+2\epsilon)n} exp\left(\sum_{l=1}^{1+\epsilon} s_l^{(n)} A_{ln}\right) - M\xi_n^{(1+\epsilon)(1+2\epsilon)}\,(s^{(n)})\right)\right.\right.$$
$$\left.\left. + exp\left(\sum_{l=1}^{1+\epsilon} s_l^{(n)} A_{ln}\right)\left(A_{(1+3\epsilon)n} - x_{(1+2\epsilon)(1+3\epsilon)} D_{(1+2\epsilon)n} + \sum_{k=1,k\neq 1+3\epsilon}^{1+4\epsilon} \alpha_k^{(n)} A_{kn}\right)\right] 1\right\|^2$$

$$= \sum_n t_n^2 \left[\|x_{(1+2\epsilon)(1+3\epsilon)}\|^2 \left\| \left(D_{(1+2\epsilon)n} exp\left(\sum_{l=1}^{1+\epsilon} s_l^{(n)} A_{ln} \right) - M\xi_n^{(1+\epsilon)(1+2\epsilon)} (s^{(n)}) \right) \right.\right.$$
$$+ exp\left(\sum_{l=1}^{1+\epsilon} s_l^{(n)} A_{ln} \right) \Bigg(A_{(1+3\epsilon)n} - x_{(1+2\epsilon)(1+3\epsilon)} D_{(1+2\epsilon)n}$$
$$\left.\left. + \sum_{k=1, k\neq 1+3\epsilon}^{1+4\epsilon} \alpha_k^{(n)} A_{kn} \Bigg) 1 \right\|^2 \right]$$
$$= \sum_n t_n^2 \left[\|x_{(1+2\epsilon)(1+3\epsilon)}\|^2 \left(c_{(1+2\epsilon)(1+2\epsilon)}^{(n)} - \left| M\xi_n^{(1+\epsilon)(1+2\epsilon)} (s^{(n)}) \right|^2 \right) \right.$$
$$+ \left\| + exp\left(\sum_{l=1}^{1+\epsilon} s_l^{(n)} A_{ln} \right) \Bigg(A_{(1+3\epsilon)n} - x_{(1+2\epsilon)(1+3\epsilon)} D_{(1+2\epsilon)n}\right.$$
$$\left.\left. + \sum_{k=1, k\neq 1+3\epsilon}^{1+4\epsilon} \alpha_k^{(n)} A_{kn} \Bigg) 1 \right\|^2 \right]$$

where we have used the equality $\|\xi - M\xi\|^2 = \|\xi\|^2 - |M\xi|^2$:

$$\# \left\| \left[D_{(1+2\epsilon)n}\, exp\left(\sum_{l=1}^{1+\epsilon} s_l^{(n)} A_{ln} \right) - M\xi_n^{(1+\epsilon)(1+2\epsilon)} s^{(n)} \right] 1 \right\|^2$$
$$= \|D_{(1+2\epsilon)n} 1\|^2 - \left| M\xi_n^{(1+\epsilon)(1+2\epsilon)} (s^{(n)}) \right|^2$$
$$= c_{(1+2\epsilon)(1+2\epsilon)}^{(n)} - \left| M\xi_n^{(1+\epsilon)(1+2\epsilon)} (s^{(n)}) \right|^2 .$$

Corollary (4.1.29)[260]: For $C \in Mat(1+2\epsilon, \mathbb{C})$ and $\lambda^2 \in \mathbb{C}^{1+2\epsilon}$ we have

$$G_{1+2\epsilon}(\lambda^2) = A_{\aleph}^{\aleph}\left(C_{1+2\epsilon}(\lambda^2)\right) = detC_{1+2\epsilon}(\lambda^2) = detC_{1+2\epsilon} + \sum_{r=1}^{1+2\epsilon} \lambda_r^2 A_r^r \left(C_{1+2\epsilon}\left(\lambda^{2[r]}\right)\right), \quad (80)$$

$$A_{1+\epsilon}^{1+\epsilon}(C_{1+2\epsilon}(\lambda^2)) = A_{1+\epsilon}^{1+\epsilon}(C_{1+2\epsilon}) + \sum_{r=1, r\neq 1+\epsilon}^{1+2\epsilon} \lambda_r^2 A_r^r \left(C_{1+2\epsilon}\left(\lambda^{2[r]}\right)\right), \quad (81)$$

$$G_{1+2\epsilon}(\lambda^2) = A_{\aleph}^{\aleph}\left(C_{1+2\epsilon}(\lambda^2)\right) = detC_{1+2\epsilon}(\lambda^2) detC_{1+2\epsilon} + \sum_{r=1}^{1+2\epsilon} \lambda_r^2 A_r^r \left(C_{1+2\epsilon}\left(\lambda^{2[r]}\right)\right) \quad (82)$$

$$A_{1+\epsilon}^{1+\epsilon}(C_{1+2\epsilon}(\lambda^2)) = A_{1+\epsilon}^{1+\epsilon}(C_{1+2\epsilon}) + \sum_{r=1, r\neq 1+\epsilon}^{1+2\epsilon} \lambda_r^2 A_{r(1+\epsilon)}^{r(1+\epsilon)} \left(C_{1+2\epsilon}\left(\lambda^{2[r]}\right)\right), \quad (83)$$

where for $\lambda^2 \in C_{1+2\epsilon}$ and $\epsilon \geq 0$ we have set

$$\lambda^{2[1+\epsilon]} = (0, \ldots, 0, \lambda_{2+\epsilon}^2, \ldots, \lambda_{1+2\epsilon}^2), \lambda^{2\{1+\epsilon\}} = (\lambda_1^2, \lambda_2^2, \ldots, \lambda_{1+\epsilon}^2, 0, \ldots, 0). \quad (84)$$

Proof: We have for $m = 2$ using (25)

$$
\begin{aligned}
G_2(\lambda^2) &= detC_2 + \lambda_1^2 A_1^1(C_2) + \lambda_2^2 A_2^2(C_2) + \lambda_1^2\lambda_2^2 A_{12}^{12}(C_2) \\
&= detC_2 + \lambda_1^2[A_1^1(C_2) + \lambda_2^2 A_{12}^{12}(C_2)] + \lambda_2^2 A_2^2(C_2) \\
&= detC_2 + \lambda_1^2 A_1^1\left(C_2\left(\lambda^{2[124]}\right)\right) + \lambda_2^2 A_2^2\left(C_2\left(\lambda^{2[125]}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
G_2(\lambda^2) &= detC_2 + \lambda_1^2 A_1^1(C_2) + \lambda_2^2[A_2^2(C_2) + \lambda_1^2 A_{12}^{12}(C_2)] \\
&= detC_2 + \lambda_1^2 A_1^1\left(C_2\left(\lambda^{2\{1\}}\right)\right) + \lambda_2^2 A_2^2\left(C_2\left(\lambda^{2\{2\}}\right)\right).
\end{aligned}
$$

For $\epsilon = 1$ we have

$$
\begin{aligned}
G_3(\lambda^2) &= detC_3 + \lambda_1^2 A_1^1(C_3) + \lambda_2^2 A_2^2(C_3) + \lambda_3^2 A_3^3(C_3) + \lambda_1^2\lambda_2^2 A_{12}^{12}(C_3) + \lambda_1^2\lambda_3^2 A_{13}^{13}(C_3) \\
&+ \lambda_2^2\lambda_3^2 A_{23}^{23}(C_3) + \lambda_1^2\lambda_2^2\lambda_3^2 A_{123}^{123}(C_3) \\
&= detC_2 + \lambda_1^2[A_1^1(C_3) + \lambda_2^2 A_{12}^{12}(C_3) + \lambda_3^2 A_{13}^{13}(C_3) + \lambda_2^2\lambda_3^2 A_{123}^{123}(C_3)] \\
&+ \lambda_2^2[A_2^2(C_3) + \lambda_3^2 A_{23}^{23}(C_3)] + \lambda_3^2 A_3^3(C_3) \\
&= detC_3 + \lambda_1^2 A_1^1\left(C_3\left(\lambda^{2[124]}\right)\right) + \lambda_2^2 A_2^2\left(C_3\left(\lambda^{2[125]}\right)\right) + \lambda_3^2 A_3^3\left(C_3\left(\lambda^{2[126]}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
G_3(\lambda^2) &= detC_3 + \lambda_1^2 A_1^1(C_3) + \lambda_2^2[A_2^2(C_3) + \lambda_1^2 A_{12}^{12}(C_3)] \\
&+ \lambda_3^2[A_1^1(C_3) + \lambda_1^2 A_{13}^{12}(C_3) + \lambda_2^2 A_{23}^{23}(C_3) + \lambda_1^2\lambda_2^2 A_{123}^{123}(C_3)] \\
&= detC_3 + \lambda_1^2 A_1^1\left(C_3\left(\lambda^{2\{1\}}\right)\right) + \lambda_2^2\left(A_2^2 C_3\left(\lambda^{2\{2\}}\right)\right) + \lambda_3^2 A_3^3\left(C_3\left(\lambda^{2\{3\}}\right)\right)
\end{aligned}
$$

For $\epsilon > 1$ the proof of (80) and (82) is the same. The identity (81) follows from (80) and (83) follows from (82).

Section (4.2): Infinite-Dimensional Groups

The induced representations were introduced and studied for finite groups by F.G.Frobenius. The aim is to develop the concept of induced representations for infinite-dimensional groups.

We devoted to the notion of induced representations elaborated for a locally compact groups by G.W.Mackey [1], [11] and to the Kirillov orbit methods [163] for the nilpotent Lie groups $B(n, \mathbb{R})$.

We extend the notion of the induced representations for infinite-dimensional groups.

We start the orbit method for infinite-dimensional “nilpotent” group $B_0^{\mathbb{Z}}$, construct the induced representations corresponding to the generic orbits and study its irreducibility.

We remind the Gauss decomposition of $n \times n$ matrices and Gauss decomposition of infinite order matrices More precisely, we give the well-known definition of the induced representations for a locally compact groups we remind the Kirillov orbit method for finite-dimensional nilpotent group $G_n = B(n, \mathbb{R})$. The induced representations, corresponding to a generic orbits of the group G_n.

We give a new proof of the irreducibility of the induced representations corresponding to a generic orbits in order to extend the proof of the irreducibility for infinite-dimensional “nilpotent” group $B_0^{\mathbb{Z}}$.

We remind the definition of the regular and quasiregular representations of infinite-dimensional groups. As in the case of a locally compact group these representations are the particular cases of the induced representations. This gives us the hint how to define the induced representations for infinite-dimensional groups. The definition is done in the questions concerning the development of the orbit method for infinite-dimensional “nilpotent” group $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$ are discussed in

The completions of the initial groups G are necessary to the definition of the induced representations for the initial infinite-dimensional group. The completions of the inductive limit $G = \lim_{\to n} G_n$ of matrix groups G_n are studied in We show that the Hilbert-Lie groups appear naturally in the representation theory of the infinite-dimensional matrix group. We define a family of the Hilbert-Lie group $GL_2(a)$ (resp. $B_2(a)$), a Hilbert completions of the group $GL_0(2\infty, \mathbb{R}) = \lim_{\to n} GL(2n-1, \mathbb{R}) (resp. B_0^{\mathbb{Z}} = -\lim_{\to n} B(2n-1, \mathbb{R}))$.

We show that any continuous representation of the group $GL_0(2\infty, \mathbb{R})$ $(resp. B_0^{\mathbb{Z}})$ is in fact continuous in some stronger topology, namely in a topology of a suitable Hilbert -Lie group $GL_2(a)$ (resp. $B_2(a)$) depending on the representation.

We construct the induced representations of the group $B_0^{\mathbb{Z}}$ corresponding to a generic orbits. The irreducibility of these representations is studied .The very first steps to describe some part of the dual for the group ℕ and $B_0^{\mathbb{Z}}$ are mentioned induced representations. The induced representation In $d_H^G S$ is the unitary representation of a group G associated with a unitary representation S : H → U(V) of a closed subgroup H of the group G. For details, see [140], Suppose that X = H \ G is a right G−space and that s : X → G is a Borel section of the projection p : G → X = HΓ: g 7→ Hg. For Lie group, such a mapping s can be chosen to be smooth almost everywhere. Then every element g ∈ G can be uniquely written in the form

$$g = hs(x), h \in H, x \in X, \tag{85}$$

and thus G (as a set) can be identified with $H \times X$. Under this identification, the Haar measure on G goes into a measure equivalent to the product of a quasi-invariant measure on X and a Haar measure on H. If a quasi-invariant measure μ_s on X is appropriately chosen, then the following equalities are valid

$$d_r(g) = \frac{\Delta_G(h)}{\Delta_H(h)} d\mu_s(x) d_r(h), \tag{86}$$

$$\frac{d\mu_s(xg)}{d\mu_s(x)} = \frac{\Delta_H(h(x,g))}{\Delta_G(h(x,g))}, \tag{87}$$

where Δ_G is a modular function on the group G and $h(x,g) \in H$ is defined by the relation

$$s(x)g = h(x,g)s(xg). \tag{88}$$

Recall that a modular function on a group G is a homomorphism $G \ni t \mapsto \Delta_G(t) \in R_+$ defined by the equality $h^{L_t} = \Delta_G(t)h$, where h is the right Haar measure on G, L is the left action of the group G on itself and $h^{L_t}(C) = h(tC)$.

Remark (4.2.1)[161]: If the group G is unimodular, i.e $\Delta_G \equiv 1$, and it is possible to select a subgroup K that is complementary to H in the sense that almost every element of G can be uniquely written in the form

$$g = hk, h \in H, k \in K, \tag{89}$$

then it is natural to identify X = H\G with K and to choose s as the embedding of K in G

$$s : K \mapsto G. \tag{90}$$

In such a case, the formula (86) assume the form

$$d\mathrm{g} = \Delta_H(h)^{-1} d_r(h) d_r(k). \tag{91}$$

If both G and H are unimodular (or, more generally, if $\Delta_G(h)$ and $\Delta_{\mathrm{H}}(h)$ coincide for h ∈ H), then there exist a G-invariant measure on X=H\G. If it is possible to extend Δ_{H} to a multiplicative function on the group G, then there exist a quasi-invariant measure on X which is multiplied by the factor $\frac{\Delta_H(g)}{\Delta_G(g)}$ under translation by g.

Now we can define In $d\,{}^{G}_{H}S$ (see [140]). Let S : H → U(V) be a unitary representation of a subgroup H of the group G in a Hilbert space V and let μ be a measure on X satisfying condition (87). Let H denote the space of all vector-valued functions f on X with values in V such that

$$\|f\|^2 := \int_X \|f(x)\|_V^2 \, d\mu(x) < \infty.$$

Let us consider the representation T given by the formula

$$[T(g)f](x) = A(x,g)f(xg) = S(h)\left(\frac{d\mu_s(xg)}{d\mu_s(x)}\right)^{1/2} f(xg), \qquad (92)$$

where

$$A(x,g) = \left[\frac{\Delta_H(h)}{\Delta_G(h)}\right]^{\frac{1}{2}} S(h), \qquad (93)$$

and where the element $h = h(x,g)$ is defined by formula (88).

Definition(4.2.2)[161]: The representation T is called the unitary induced representation and is denoted by $\mathrm{Ind}_H^G\, S$.

Orbit method for finite-dimensional nilpotent group $B(n,\mathbb{R})$. See Kirillov [139] and [140]. ”Fix the group $G_n = B(n,\mathbb{R})$ of all upper triangular real matrices of order n with ones on the main diagonal. (The Kirillov notation for the group $B(n,\mathbb{R})$ is $N + (n,\mathbb{R})$).

The basic result of the method of orbits, applied to nilpotent Lie groups, is the description of a one-to-one correspondence between two sets:

(a) the set $\hat{G}$ of all equivalence classes of irreducible unitary representations of a connected and simply connected nilpotent Lie group G,

(b) the set $O(G)$ of all orbits of the group G in the space g^* dual to the Lie algebra g with respect to the coadjoint representation.

To construct this correspondence, we introduce the following definition. A subalgebra $h \subset g$ is subordinate to a functional $f \in g^*$ if

$$\langle f, [x,y]\rangle = 0 \; for\ all\ x, y \in h,$$

i.e. if h is an isotropic subspace with respect to the bilinear form defined by $B_f(x,y) = \langle f, [x,y]\rangle$ on g.

Lemma(4.2.3)[161]: (Lemma 7.7, [140]). The following conditions are equivalent:

(a) a subalgebra h is subordinate to the functional f,

(b) the image of h in the tangent space $T_f\Omega$ to the orbit Ω in the point f is an isotropic subspace,

(c) the map

$$x \mapsto \langle f, x\rangle$$

is a one-dimensional real representation of the Lie algebra h.

If the conditions of Lemma **(**4.2.3) are satisfied, we define the one-dimensional unitary representation $U_{f,H}$ of the group $H = exp\, h$ by the formula

$$U_{f,H}(exp\, x) = exp\, 2\pi i\langle f, x\rangle.$$

Theorem(4.2.4)[161]:(Theorem 7.2, [140]). (a) Every irreducible unitary representation T of a
connected and simply connected nilpotent Lie group G has the form

$$T = \mathrm{Ind}_H^G\, U_{f,H},$$

where H ⊂ G is a connected subgroup and $f \in g^*$;
(b) the representation $T_{f,H} = \mathrm{Ind}_H^G U_{f,H}$ is irreducible if and only if the Lie algebra h of the group H is a subalgebra of g subordinate to the functional f with maximal possible dimension;
(c) irreducible representations T_{f_1,H_1} and T_{f_2,H_2} are equivalent if and only if the functional f_1 and f_2 belong to the same orbit of g^*."

Example(4.2.5)[161]: Let us consider the Heisenberg group $G_3 = B(3,\mathbb{R})$, its Lie algebra g and the dual space g^*. Fix the notations

$$g = B(3,\mathbb{R}) = \left\{\begin{pmatrix}1 & x_{12} & x_{13}\\ 0 & 1 & x_{23}\\ 0 & 0 & 1\end{pmatrix}\right\},$$

$$g = B(3,\mathbb{R}) = \left\{\begin{pmatrix}1 & x_{12} & x_{13}\\ 0 & 1 & x_{23}\\ 0 & 0 & 1\end{pmatrix}\right\}, g* = n-(3,\mathbb{R}) = \left\{\begin{pmatrix}0 & 0 & 0\\ y_{21} & 0 & 0\\ y_{31} & y_{32} & 0\end{pmatrix}\right\}.$$

The adjoint action Ad : G → Aut(g) of the group G on its Lie algebra g is:

$$g \ni x \mapsto Ad_t(x) := txt^{-1} \in g, t \in G, \tag{94}$$

the pairing between the g and g^*:

$$g^* \times g \ni (y,x) \mapsto \langle y,x\rangle := tr(xy) = \sum_{1\le k<n\le 3} x_{kn}y_{nk} \in \mathbb{R}. \tag{95}$$

Since $tr(txt^{-1}y) = tr(xt^{-1}yt)$ the coadjoint action of G on the dual g^* to g is

$$g^* \ni y \mapsto Ad_t^*(y) := (t^{-1}yt)- \in g^*, t \in G, \tag{96}$$

where (z)– means that we take lower triangular part of the matrix z.
To calculate $Ad_t^*(y)$ explicitly for n = 3, we have

$$t^{-1}yt = \begin{pmatrix}1 & x_{12} & x_{13}\\ 0 & 1 & x_{23}\\ 0 & 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}0 & 0 & 0\\ y_{21} & 0 & 0\\ y_{31} & y_{32} & 0\end{pmatrix}\begin{pmatrix}1 & x_{12} & x_{13}\\ 0 & 1 & x_{23}\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & t_{12} & x_{13}+t_{12}t_{23}\\ 0 & 1 & -t_{23}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ y_{21} & y_{21}t_{12} & y_{21}t_{13}\\ y_{31} & y_{31}t_{12}+y_{32} & y_{31}t_{13}+y_{32}t_{23}\end{pmatrix},$$

hence

$$Ad_t^*(y) := (t^{-1}yt)_- = \begin{pmatrix}0 & 0 & 0\\ y_{21-t_{23}y_{31}} & 0 & 0\\ y_{31} & y_{31}t_{12}+y_{32} & 0\end{pmatrix}.$$

We have two type of the orbits O:
(I) if $y_{31} = 0$, then $\begin{pmatrix}y_{21} & \\ 0 & y_{32}\end{pmatrix} \simeq (y_{21}, y_{32})$ for fixed y_{21}, y_{32} is 0-dimensional orbit;
(II) if $y_{31} \neq 0$, then $\begin{pmatrix}\mathbb{R} & \\ y_{31} & \mathbb{R}\end{pmatrix}$ is 2-dimensional orbits.

In the case (I) fixe the point $f = (y_{21}, y_{32})$, the subordinate subalgebra h coinside with all g , since $[g, g] = \langle E_{13}\rangle := \{tE_{13} \mid t \in \mathbb{R}\}$. Corresponding one-dimensional representation of the algebra $h = g$ is

$$g \ni x \mapsto \langle f, x\rangle = tr(xf) = tr\left[\begin{pmatrix}1 & x_{12} & x_{13}\\ 0 & 0 & x_{23}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ y_{21} & 0 & 0\\ y_{31} & y_{32} & 0\end{pmatrix}\right]$$

$$= x_{12}y_{21} + x_{23}y_{32} \in \mathbb{R}.$$

The corresponding representation of the group G is

$$G \ni exp(x) \mapsto exp(2\pi_i\langle f, x\rangle) \in S^1. \quad (97)$$

So we have 1-dimensional representation

$$G_3 \ni exp\begin{pmatrix}1 & x_{12} & x_{13}\\ 0 & 0 & x_{23}\\ 0 & 0 & 1\end{pmatrix} \longmapsto exp(2\pi_i(x_{12}y_{21} + x_{23}y_{32})) \in S^1.$$

We note that

$$exp(x) = \begin{pmatrix}1 & x_{12} & x_{13}\\ 0 & 0 & x_{23}\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & x_{12} & x_{13} + \frac{1}{2}x_{12}x_{23}\\ 0 & 0 & x_{23}\\ 0 & 0 & 1\end{pmatrix}.$$

In the case 2) we have two subordinate subalgebras of the maximal dimension

$$h_1 = \begin{pmatrix}0 & 0 & x_{13}\\ 0 & 0 & x_{23}\\ 0 & 0 & 0\end{pmatrix}, and\ h_2 = \begin{pmatrix}0 & x_{12} & x_{13}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. Set\ f = \begin{pmatrix}0 & 0 & 0\\ y_{21} & 0 & 0\\ y_{31} & x_{13} & 0\end{pmatrix}.$$

The corresponding one-dimensional representations of the subalgebras $h_i, i = 1, 2$ are

$$h_1 \ni x \mapsto \langle f, x\rangle = x_{13}y_{31} + x_{23}y_{32} \in \mathbb{R},$$
$$h_2 \ni x \mapsto \langle f, x\rangle = x_{12}y_{21} + x_{13}y_{31} \in \mathbb{R}.$$

The corresponding representations S of the subgroups H_1 and H_2 respectively are:

$$H_1 \ni \begin{pmatrix}1 & 0 & x_{13}\\ 0 & 1 & x_{23}\\ 0 & 0 & 1\end{pmatrix} = exp(x) \mapsto exp(2\pi i(x_{13}y_{31} + x_{23}y_{32})) \in S^1,$$

$$H_2 \ni \begin{pmatrix}1 & x_{12} & x_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = exp(x) \mapsto exp(2\pi i(x_{12}y_{21} + x_{13}y_{31})) \in S^1$$

In the case H_1 we have the decomposition $G_3 = \mathbb{R}^2 \ltimes B(2, \mathbb{R}) \simeq H_1 \ltimes \mathbb{R}$, indeed we have

$$G_3 \ni \begin{pmatrix}1 & x_{12} & x_{13}\\ 0 & 1 & x_{23}\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & x_{13}\\ 0 & 1 & x_{23}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x_{12} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in \mathbb{R}^2 \ltimes B(2, \mathbb{R}),$$

hence the space $X = H_1\backslash G_3$ is isomorphic to $B(2, \mathbb{R}) \simeq \mathbb{R}$ and s can be choosing as the embedding $s : B(2, \mathbb{R}) \to B(3, \mathbb{R})$.

$$B(2, R) \ni (\begin{matrix}1 & x\\ 0 & 1\end{matrix}) =: x \mapsto s(x) = \begin{pmatrix}1 & x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \in B(3, \mathbb{R}).$$

For general n we have

$$B(n + 1\mathbb{R}) = \mathbb{R}^2 \ltimes B(n, \mathbb{R}). \quad (98)$$

To calculate the right action of G on X i.e. to find h(x, t) such that

$$s(x)t = h(x, t)s(xt),$$

we have for $x \in B(2, \mathbb{R})$ and $t \in B(3, \mathbb{R})$

$$s(x)t = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + t_{12} & x_{13} + xt_{23} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & x_{13} + xt_{23} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x + t_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= h(x,t)s(xt), \text{hence } h(x,t) = \begin{pmatrix} 1 & 0 & x_{13} + xt_{23} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

Finally, the induced unitary representation $\mathrm{Ind}_{H_1}^{G} S$ have the following form in the Hilbert space $L^2(\mathbb{R}, dx)$ $(case\ H_1\ and\ f = y_{31}E_{31})$:

$$f(x) \mapsto S\big(h(x,t)\big)f(xt) = exp(2\pi i(t_{13} + t_{23}x)y_{31})f(x + t_{12}). \tag{99}$$

In the Kirillov [140] notations we have:

$$f(x) \to exp(2\pi i(c + bx) \leftthreetimes)f(x + a), y_{31} = \leftthreetimes, \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

We show following A. Kirillov [140] how the orbit method works for the nilpotent group $\mathrm{B}(\mathrm{n}, \mathbb{R})$ and small n.

For general $\mathrm{n} \in \mathbb{N}$ the coadjoint action of the group $\mathrm{G_n}$ on g is as follows

$$\mathrm{t} = \mathrm{I} + \sum_{1 \le \mathrm{k} < m \le \mathrm{n}} \mathrm{t_{km}} \mathrm{E}_{\mathrm{km}}, \mathrm{y} = \sum_{1 \le \mathrm{m} < k \le \mathrm{n}} \mathrm{t_{km}} \mathrm{E}_{\mathrm{km}}, t^{-1} := \mathrm{I} + \sum_{1 \le \mathrm{k} < m \le \mathrm{n}} \mathrm{t_{km}^{-1}} \mathrm{E_{km}}$$

hence

$$(tyt^{-1})_{pq} = \sum_{m=1}^{q} (ty)_{pm} t_{mq}^{-1} = \sum_{m=1}^{q} \sum_{r=p}^{n} t_{pr} y_{rm} t_{mq,}^{-1}\ 1 \le p, q \le n,$$

and

$$Ad_t^*(y) = (t^{-1}yt)_{-} = I + \sum_{1 \le q < p \le n} (t^{-1}yt)_{pq} E_{pq}. \tag{100}$$

Example (4.2.6)[161]: Generic orbits for the group $G = B(n, \mathbb{R})$ (see [140], Example 7.9). "The form of the action $Ad_t^*(y) = (t - 1yt)_{-}$ implies, that $Ad_t^*, t \in G$ acts as follows: to a given column of $y \in g^*$, a linear combination of the previous columns is added and to a given row of y, a linear combination of the following rows is added. More generally, the minors $\Delta_k, k = 1, 2, \dots, [\frac{n}{2}]$, consisting of the last k rows and first k columns of y are invariant of the action. It is possible to show that if all the numbers c_k are different from zeros, then the manifold given by the equation

$$\Delta_k = c_k, \qquad 1 \le k \le \left[\frac{n}{2}\right] \tag{101}$$

is a G-orbit in g^*. Hence generic orbits have codimension equal to $\left[\frac{n}{2}\right]$ and dimension equal to $\frac{n(n-1)}{2} - [\frac{n}{2}]$. To obtain a representation for such an orbit, we can take a matrix y of the form

$$y = \begin{pmatrix} 0 & 0 \\ \Lambda & 0 \end{pmatrix},$$

where Λ is the matrix of order $[\frac{n}{2}]$ such that all nonzero elements are contained in the anti-diagonal. It is easy to find a subalgebra of dimension $[\frac{n}{2}] \times [\frac{n+1}{2}]$ subordinate to the functional y. It consist of all matrices of the form

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

where A is an $[\frac{n}{2}] \times [\frac{n+1}{2}] or [\frac{n+1}{2}] \times [\frac{n}{2}]$ matrix."

Example(4.2.7)[161]: Let $G = B(5,\mathbb{R}), g = n_+(5,\mathbb{R}), g^* = n_-(5,\mathbb{R})$. We write the repre-sentations for generic orbit corresponding to the point $y = y_{51}E_{51} + y_{42}E_{42} \in g^*$. Set $h_3 = \{t^{-I} \mid t \in H_3\}$ where

$$G = \left\{\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 1 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}, H_3 = \left\{\begin{pmatrix} 1 & 0 & 0 & t_{14} & t_{15} \\ 0 & 1 & 0 & t_{24} & t_{25} \\ 0 & 0 & 1 & t_{34} & t_{35} \\ 0 & 0 & 0 & 1 & 0 \\ `0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

$$= g^* \left\{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 & 0 \\ y_{31} & y_{32} & 0 & 0 & 0 \\ y_{41} & y_{42} & y_{43} & 0 & 0 \\ y_{51} & y_{52} & y_{53} & y_{54} & 0 \end{pmatrix}\right\}$$

The corresponding representation S of the subgroup H_3 of the maximal dimension is:

$$H_3 \ni t \mapsto exp(2\pi i\langle y,(t - I)\rangle) = exp(2\pi i[t_{15}y_{51} + t_{24}y_{42}]) \in S^1.$$

For the group $B(5,\mathbb{R})$ holds the following decomposition

$$B(5,\mathbb{R}) = B_3B(3)B^{(3)} \, i.e. \, x = x_3x(3)x^{(3)}, \tag{102}$$

where

$$B^{(3)} = \left\{\begin{pmatrix} 1 & x_{12} & x_{13} & 0 & 0 \\ 0 & 1 & x_{23} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}, B(3) = \left\{\begin{pmatrix} 1 & 0 & 0 & t_{14} & t_{15} \\ 0 & 1 & 0 & t_{24} & t_{25} \\ 0 & 0 & 1 & t_{34} & t_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

$$= B_3 \left\{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_{54} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

We calculate h(x, t) in the relation $s(x)t = h(x,t)s(xt)$, but first we fix the section $s : X = H\backslash G \mapsto G$ of the projection p : G → X. To define the section $s : X \mapsto G$ we show that in addition to the decomposition (102) the following decomposition $B(5,\mathbb{R}) = B(3)B_3B^{(3)}$ also holds. Indeed, to find $h \in H_3 = B^{(3)}$ such that $x = hx_3x^{(3)}$, we get $x_3x(3)x^{(3)} = hx_3x^{(3)}$, hence

$$h = x_3 x(3) x_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & x_{14} & x_{15} \\ 0 & 1 & 0 & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & x_{14} & x_{15} - x_{14}x_{45} \\ 0 & 1 & 0 & x_{24} & x_{25} - x_{24}x_{45} \\ 0 & 0 & 1 & x_{34} & x_{35} - x_{34}x_{45} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in B(3).$$
We have two different decompositions
$$B_3 B(3) B^{(3)} \ni x_3 x(3) x^{(3)} = h x_3 x^{(3)} \in B(3) B_3 B^{(3)}, with\, h = x_3 x(3) x_3^{-1}.$$
Remark (4.2.8)[161]: For an arbitrary $n, m \in \mathbb{N}, 1 < m < n$, we have for the group $G_n = B(n, \mathbb{R})$ $t$wo decompositions:

$$G_n = B_m B(m) B^{(m)} \ni x_m x(m) x^{(m)} = h x_m x^{(m)} \in B(m) B_m B^{(m)}, h = x_m x(m) x_m^{-1}, (103)$$
where
$$B_m = \{I + \sum_{m<k<r\leq n} x_{kr} E_{kr}\}, B(m) = \{I + \sum_{1\leq k\leq m<r\leq n} x_{kr} E_{kr}\}, B^{(m)} = \{I + \sum_{1\leq k<r\leq m} x_{kr} E_{kr}\}.$$
Since $X = B(m) \backslash G_n$ is isomorphic to $B_m B^{(m)}$ by decomposition (103), the section scan be choosing, by Remark**(**4.2.1) , as the embedding
$$B_m B^{(m)} \ni x_m x^{(m)} \mapsto s(x_m x^{(m)}) = x_m x^{(m)} \in B_m B(m) B^{(m)}.$$
Since $s(x)t = h(x,t)s(xt)$, we have $h(x,t) = s(x)t(s(xt))^{-1}$. It remains to calculate $s(x)t$ and $s(xt)$.

Remark(4.2.9)[161]: We have
$$h(x,t) - I = \begin{cases} 0, for\ t \in B_m B^{(m)} \\ x^{(m)}(t - I)x_m^{-1}, for\ t \in B(m) \end{cases}.$$
Indeed, let $t = t_m t(m) \in B_m B^{(m)}$. then $s(x)t = x_m x^{(m)} t_m t^{(m)} = x_m t_m x^{(m)} t^{(m)}$ We get also $xt = x_m x^{(m)} t_m t^{(m)} = x_m t_m x^{(m)} t^{(m)}, so\ s(xt) = x_m t_m x^{(m)} t^{(m)}$, hence $s(x)t = s(xt)$ and we get $h(x,t) = e$. For $t := t(m) \in B(m)$ and $x = x_m x^{(m)} \in B_m B(m)$ we get
$$s(x)t = x_m x^{(m)} t = x_m x^{(m)} t (x^{(m)})^{-1} x^{(m)} = x_m \tilde{x}(m) x^{(m)} = h x_m x^{(m)} = h(x,t)s(xt),$$
where $\tilde{x}(m) = x^{(m)} t(x(m))^{-1}$. Then we get by (103)
$$h(x,t) = h = x_m \tilde{x}(m) x_m^{-1} = x_m x^{(m)} t \left(x^{(m)}\right)^{-1} x_m^{-1} = x_m x^{(m)} t \left(x_m x^{(m)}\right)^{-1}, \quad (104)$$
$$h(x,t) = \begin{pmatrix} x^{(m)} & 0 \\ 0 & x_m \end{pmatrix} \begin{pmatrix} 1 & t - I \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \left(x^{(m)}\right)^{-1} & 0 \\ 0 & x_m^{-1} \end{pmatrix} = \begin{pmatrix} 1\ x^{(m)} & (t - I)x_m^{-1} \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & H(x,t) \\ 0 & 1 \end{pmatrix}, \quad (105)$$

where

$$(x,t) := x^{(m)}(t - I)x_m^{-1} . \qquad (106)$$

Denote by $E_{kr}(t) := I + tE_{kr}, t \in \mathbb{R}$ the one-parameter subgroups of the groups $B(n,\mathbb{R})$. We would like to find the generators $A_{kn} = \frac{d}{dt} T_{I+tE_{kn}}|_{t=0}$ of the induced representation T_t (112).

Set for $G_n = B_m B(m) B^{(m)}$ and $1 \le k \le m < r \le n$

$$S_{kr}(t_{kr}) := \langle y, \left(h\left(x, E_{kr}(t_{kr})\right) - I\right)\rangle,$$

then

$$A_{kr} = \frac{d}{dt} exp\left(2\pi i S_{kr}(t)\right)|_{t=0} = 2\pi i S_{kr}(1). \qquad (107)$$

Let us denote by $\mathbb{S}$ the following matrix:

$$\mathbb{S} = (S_{kr})_{1\le k\le m<r\le n,} \text{ } where \text{ } S_{kr} = S_{kr}(1), then \text{ } \mathbb{S} = (2\pi i)^{-1}(A_{kr})_{k,r}. \qquad (108)$$

Lemma(4.2.10)[161]: Let $B = (b_{kr})_{k,r=1}^{n} \in Mat(n,\mathbb{C})$. Define the matrix $C = (c_{kr})_{k,r=1}^{n} \in Mat(n,\mathbb{C})$ by

$c_{kr} = tr(E_{kr}B), 1 \le k,r \le n,$ then we have $= B^T$, (109)

where E_{kr} are matrix units and B^T means transposed matrix to the matrix B. The equality $C = B^T$ holds also in the case when B is an arbitrary $m \times n$ rectangular matrix. The statement is true also for matrices $B \in Mat(\infty,\mathbb{C})$.

Proof. Indeed, we have $tr(E_{kr}B) = b_{rk}$.

We calculate now the matrix $\mathbb{S}(t) = (S_{kr}(t_{kr}))_{k,r}$ and the matrix $\mathbb{S} = (S_{kr}(1))_{k,r}$ using Lemma(4.2.10) Using (106) we have

$\langle y, h(x,t) - I \rangle = tr\ (H(x,t)y) = tr\left((m)t_0 x_m^{-1}\ y\right) = tr\left(t_0 x_m^{-1}\ y x^{(m)}\right) =$

$tr\ (t_0 B(x,y))$, where $t_0 = t - I$ and

$$B(x,y) = x_m^{-1}\ y x^{(m)} \cong \begin{pmatrix} 1 & 0 \\ 0 & x_m^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x_m^{-1} y x^{(m)} & 0 \end{pmatrix}. \qquad (110)$$

By definition we have

$$S_{kr}(t_{kr}) = \langle y, (h(x, E_{kr}(t_{kr})) - I)\rangle = tr(t_{kr}E_{kr}B(x,y)),$$

hence by Lemma(4.2.10) and (4.2.3) (110) we conclude that

$$\mathbb{S} = (S_{kr}(1))_{kr} = (tr\ (E_{kr}B(x,t)))_{k,r} = B^T\ (x,y) = (x^{(m)})^T\ y^T\ (x_m^{-1}\ )^T$$

$$= \begin{pmatrix} 0 & (x^{(m)})^T\ y^T\ (x_m^{-1}\ )^T \\ 0 & 0 \end{pmatrix}. \qquad (111)$$

So the induced representation $\text{Ind}_H^G(S) : G \to U(L^2(X,\mu))$ corresponding to the point $y \in g^*$ has the following form

$$(T_t f)(x) = S(h(x,t))\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{\frac{1}{2}} f(xt), f \in L^2(X,\mu), x \in X = H\backslash G, t \in G, \qquad (112)$$

where

$$\mathbb{S}(h(x,t)) = exp(2\pi i\langle y, (h(x,t) - I)\rangle) = exp 2\pi i tr\ ((t - I)B(x,y)). \qquad (113)$$

We calculate B(x, y) and S for different groups G_n. For G_5 we get by (110):

$$G5 = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 1 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & y_{42} & 0 & 0 & 0 \\ y_{51} & 0 & 0 & 0 & 0 \end{pmatrix}, x^{(3)}$$

$$= \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix}, x_3 = \begin{pmatrix} 1 & x_{45} \\ 0 & 1 \end{pmatrix},$$

$$B(x,y) = \begin{pmatrix} 1 & x_{45}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & y_{24} & 0 \\ y_{25} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_{45}^{-1} y_{51} & y_{42} + x_{45}^{-1} y_{51} x_{12} & y_{42} x_{23} + x_{45}^{-1} y_{51} x_{13} \\ y_{51} & y_{51} x_{12} & y_{51} x_{13} \end{pmatrix},$$

hence by (111) we have

$$\mathbb{S} := B(x,y)^T = \begin{pmatrix} 1 & x_{45}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & y_{42} & 0 \\ y_{51} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_{45}^{-1} y_{51} & y_{51} \\ y_{42} + x_{45}^{-1} y_{51} x_{12} & y_{51} x_{12} \\ y_{42} x_{23} + y_{51} x_{12} & y_{51} x_{13} \end{pmatrix} \tag{114}$$

Remark(4.2.11)[161]: For the matrix $x = I + \sum_{1 \le k < n \le m} x_{kn} E_{kn} \in B(m, \mathbb{R})$ we denote by x_{kn}^{-1} the matrix elements of the matrix $x^{-1}, i.e. x^{-1} =: I + \sum_{1 \le k < n \le m} x_{kn}^{-1} E_{kn} \in B(m, \mathbb{R})$.

The explicit expressions for x_{kn}^{-1} are as follows (see [165]) $x_{kk+1}^{-1} = -x_{kk+1}$,

$$x_{kn}^{-1} = -x_{kn} + \sum_{r=1}^{n-k-1} (-1)^{r-1} \sum_{k < i_1 < i_2 < \dots < i_r < n} x_{k i_1} x_{i_1 i_2} \dots x_{i_r n}, k < n - 1. \tag{115}$$

The generators $A_{kn} = \frac{d}{dt} T_{I + tE_{kn}}|_{t=0}$ of the one-parameter subgroups $E_{kn}(t) := I + tE_{kn}, t \in \mathbb{R}$ generated by the representation T_t (112) are as follows (see (108) and(114)):

$$A_{12} = D_{12}, A_{13} = D_{13}, \quad A_{23} = x_{12} D_{13} + D_{23}, \quad A_{45} = D_{45}, \tag{116}$$

$$S = \frac{1}{2\pi i} \begin{pmatrix} A_{14} & A_{15} \\ A_{24} & A_{25} \\ A_{34} & A_{35} \end{pmatrix} = \begin{pmatrix} x_{45}^{-1} y_{51} & y_{51} \\ y_{42} + x_{45}^{-1} y_{51} x_{12} & y_{51} x_{12} \\ y_{42} x_{23} + x_{45}^{-1} y_{51} x_{12} & y_{51} x_{13} \end{pmatrix} \tag{117}$$

Where $D_{kn} = \frac{\partial}{\partial x_{kn}}$ For example, to obtain the expression $A_{23} = x_{12} D_{13} + D_{23}$ we notethat

$$B(3, \mathbb{R}) \ni x(I + tE_{23}) = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_{12} & x_{13} + t x_{12} \\ 0 & 1 & x_{23} + t \\ 0 & 0 & 1 \end{pmatrix}$$

Here we denote by $kn = D_{kn}(h)$ the operator of the partial derivative corresponding to the shift $x \mapsto x + tE_{kn}$ on the group $B_m \times B^{(m)} \ni x = (x_{kn})_{k,n}$ and the Haar measure h:

$$(D_{kn}(h) f)(x) = \frac{d}{dt} \left(\frac{dh(x + tE_{kn})}{dh(x)} \right)^{\frac{1}{2}} f(x + tE_{kn}) |_{t=0}, D_{kn}(h) := \frac{\partial}{\partial x_{kn}} \tag{118}$$

Example(4.2.12)[161]: Let $G = B(4,R) = \begin{Bmatrix} 1 & x_{23} & x_{24} & x_{25} \\ 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 1 & x_{25} \\ 0 & 0 & 0 & 1 \end{Bmatrix}$.The representations for genericorbit corresponding to the point $y = y_{43}E_{\ 43} + y_{52}E_{52} \in g^*$.
We calculate S in two different ways. First using (110) we get

$$B(x,y) = x_m^{-1}\,yx^{(m)} = \begin{pmatrix} 1 & x_{45}^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y_{43} \\ y_{52} & 0 \end{pmatrix}\begin{pmatrix} 1 & x_{23} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{45}^{-1}y_{52} & y_{43} + x_{45}^{-1}x_{23} \\ y_{52} & x_{23}y_{52} \end{pmatrix},$$

$$\frac{1}{2\pi i}\begin{pmatrix} A_{24} & A_{25} \\ A_{34} & A_{34} \end{pmatrix} = \mathbb{S} = B^T(x,y) = \begin{pmatrix} 1 & 0 \\ x_{23} & 1 \end{pmatrix}\begin{pmatrix} 0 & y_{52} \\ y_{43} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_{45}^{-1} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} x_{45}^{-1}y_{52} & y_{52} \\ y_{43} + x_{45}^{-1}y_{52}x_{23} & y_{52}x_{23} \end{pmatrix}$$
$$A_{23} = D_{23},\ \ A_{45} = D_{45}$$

From the other hand, by (105) we get $h(x,t) = \begin{pmatrix} 1 & H(x,t) \\ 0 & 1 \end{pmatrix}$, where

$$H(x,t) = x^{(3)}(t-I)x_3^{-1} = \begin{pmatrix} 1 & x_{23} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t_{24} & t_{25} \\ t_{34} & t_{35} \end{pmatrix}\begin{pmatrix} 1 & x_{45}^{-1} \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} t_{24} + x_{23}t_{34} & (t_{24} + x_{23}t_{34})x_{45} + t_{25} + x_{23}t_{35} \\ t_{34} & t_{34}x_{45}^{-1} + t_{25} + t_{35} \end{pmatrix} \tag{119}$$

Therefore,

$$\langle y, (h(x,t) - I)\rangle = h(x,t)_{34}y_{43} + h(x,t)_{25}y_{52}$$
$$= t_{34}y_{43} + [(t_{24} + x_{23}t_{34})x_{45}^{-1} + t_{25} + x_{23}t_{35}]y_{52,}$$

Hence

$$\mathbb{S}_2\begin{pmatrix} s_{24}(t_{24}) & s_{25}(t_{25}) \\ s_{34}(t_{34}) & 5_{35}(t_{35}) \end{pmatrix} = (t) := \begin{pmatrix} t_{24}x_{45}^{-1}y_{52} & t_{25}y_{52} \\ t_{34}y_{43} + x_{23}t_{34}x_{45}^{-1}y_{52} & x_{23}t_{35}y_{52} \end{pmatrix},$$
$$\mathbb{S}_2 := \mathbb{S}_2(1)\begin{pmatrix} s_{24} & s_{25} \\ s_{34} & 5_{35} \end{pmatrix} = \begin{pmatrix} x_{45}^{-1}y_{52} & y_{52} \\ y_{43} + x_{45}^{-1}y_{52}x_{23} & y_{52}x_{23} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ x_{23} & 1 \end{pmatrix}\begin{pmatrix} 0 & y_{25} \\ y_{43} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_{45}^{-1} & 1 \end{pmatrix} \tag{120}$$

Example (4.2.13)[161]: Let $G = B(6,\mathbb{R}), g = n_+(6,\mathbb{R}), g^* = n_-(6,\mathbb{R})$. We write the representations for generic orbit corresponding to the point $y = y_{43}E_{43} + y_{52}E_{52} + y_{61}E_{61} \in g^*$.Set

$$G_6 = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ 0 & 1 & x_{23} & x_{24} & x_{25} & x_{26} \\ 0 & 0 & 1 & x_{34} & x_{35} & x_{36} \\ 0 & 0 & 0 & 1 & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & 1 & x_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \right\}, H_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 & x_{14} & x_{15} & x_{16} \\ 0 & 1 & 0 & x_{24} & x_{25} & x_{26} \\ 0 & 0 & 1 & x_{34} & x_{35} & x_{36} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{43} & 0 & 0 & 0 \\ 0 & y_{52} & 0 & 0 & 0 & 0 \\ y_{51} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$h_3 = \{t - I \mid t \in H_3\}$. The corresponding representations S of the subgroup H_3 is:
$H_3 \ni exp(t - I) = t \mapsto exp(2\pi i\langle y, (t - I)\rangle) = exp(2\pi i[t_{34}y_{43} + ty52 + t_{16}y_{61}]) \in S^1$.
For the group B(6,R) holds the following decomposition

$$B(6,\mathbb{R}) = B_3 B(3) B^{(3)} i.e. x = x_3 x(3) x^{(3)}, \qquad (121)$$

where

$$x^{(3)} = \begin{pmatrix} 1 & x_{12} & x_{13} & 0 & 0 & 0 \\ 0 & 1 & x_{23} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad x(3) = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ 0 & 1 & x_{23} & x_{24} & x_{25} & x_{26} \\ 0 & 0 & 1 & x_{34} & x_{35} & x_{36} \\ 0 & 0 & 0 & 1 & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & 1 & x_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & 1 & x_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We get by (110) and (111)

$$B(x,y) = \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} \\ 0 & 1 & x_{56}^{-1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & y_{43} \\ 0 & y_{52} & 0 \\ y_{61} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_{46}^{-1}y_{61} & x_{45}^{-1}y_{52} + x_{46}^{-1}y_{61}x_{12} & y_{43} + x_{45}^{-1}y_{52}x_{23} + x_{46}^{-1}y_{61}x_{13} \\ x_{56}^{-1}y_{61} & y_{52} + x_{56}^{-1}y_{61}x_{12} & y_{52}x_{23} + x_{56}^{-1}y_{61}x_{13} \\ y_{61} & y_{61}x_{12} & y_{61}x_{13} \end{pmatrix},$$

hence

$$\mathbb{S} = B^T(x,y) = \begin{pmatrix} 1 & 0 & 0 \\ x_{12} & 1 & 0 \\ x_{13} & x_{23} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & y_{61} \\ 0 & y_{52} & 0 \\ y_{43} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x_{45}^{-1} & 1 & 0 \\ x_{46}^{-1} & x_{56}^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_{46}^{-1}y_{61} & x_{56}^{-1}y_{61} & y_{61} \\ x_{45}^{-1}y_{52}x_{46}^{-1}y_{61}x_{12} & y_{52}x_{46}^{-1}y_{61}x_{12} & y_{61}x_{12} \\ y_{43}+x_{45}^{-1}y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13} & y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13} & y_{61}x_{13} \end{pmatrix}$$

Using again (108), (112) and Remark **(**4.2.9) we get the following expressions for the generators $A_{kn} = \frac{d}{dt} T_{I+tE_{kn}}|_{t=0}$ of one-parameter subgroups $I + tE_{kn}, t \in \mathbb{R}$:

$$A_{12} = D_{12}, A_{13} = D_{13}, A_{23} = x_{12}D_{13} + D_{23}, \tag{122}$$

$$A_{45} = D_{45}, A_{64} = D_{46}, A_{56} = x_{45}D_{46} + D_{56}, \tag{123}$$

$$\mathbb{S} = \frac{1}{2\pi i}\begin{pmatrix} A_{14} & A_{15} & A_{16} \\ A_{24} & A_{25} & A_{26} \\ A_{34} & A_{35} & A_{36} \end{pmatrix} = \begin{pmatrix} x_{46}^{-1}y_{61} & x_{56}^{-1}y_{61} & y_{61} \\ x_{45}^{-1}y_{52}x_{46}^{-1}y_{61}x_{12} & y_{52}x_{46}^{-1}y_{61}x_{12} & y_{61}x_{12} \\ y_{43}+x_{45}^{-1}y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13} & y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13} & y_{61}x_{13} \end{pmatrix} \tag{124}$$

We recall the expressions for $B(x,y)$ and hence for $\mathbb{S} = B(x,y)^T$ for small n. For n = 4we have

$$B(x,y) = x_m^{-1}\, yx^{(m)} = \begin{pmatrix} 1 & x_{45}^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & y_{43} \\ y_{52} & 0 \end{pmatrix}\begin{pmatrix} 1 & x_{23} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{45}^{-1}y_{52} & y_{43}+x_{45}^{-1}y_{52}x_{23} \\ y_{52} & y_{52}x_{23} \end{pmatrix}$$

$$\mathbb{S} = \begin{pmatrix} 1 & 0 \\ x_{23} & 1 \end{pmatrix}\begin{pmatrix} 0 & y_{52} \\ y_{43} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_{45}^{-1} & 1 \end{pmatrix} = \begin{pmatrix} x_{45}^{-1}y_{52} & y_{52} \\ y_{43}+x_{45}^{-1}y_{52}x_{23} & y_{52}x_{23} \end{pmatrix}.$$

For $G_2^3 \simeq B(6,\mathbb{R})$ (see (2.41) for the notation G_n^m) holds:

$$B(x,y) = \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} \\ 0 & 1 & x_{46}^{-1} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & y_{43} \\ 0 & y_{52} & 0 \\ y_{61} & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_{46}^{-1}y_{61} & x_{45}^{-1}y_{52}+x_{46}^{-1}y_{61}x_{12} & y_{43}+x_{45}^{-1}y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13} \\ x_{56}^{-1}y_{61} & y_{52}+x_{56}^{-1}y_{61}x_{12} & y_{52}x_{23}+x_{56}^{-1}y_{61}x_{13} \\ y_{61} & y_{61}x_{12} & y_{61}x_{13} \end{pmatrix},$$

Hence

$$\mathbb{S} = \begin{pmatrix} x_{46}^{-1}y_{61} & x_{56}^{-1}y_{61} & y_{61} \\ x_{45}^{-1}y_{52}x_{46}^{-1}y_{61}x_{12} & y_{52}x_{46}^{-1}y_{61}x_{12} & y_{61}x_{12} \\ y_{43}+x_{45}^{-1}y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13} & y_{52}x_{23}+x_{46}^{-1}y_{61}x_{13} & y_{61}x_{13} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ x_{12} & 1 & 0 \\ x_{13} & x_{23} & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & y_{61} \\ 0 & y_{52} & 0 \\ y_{43} & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ x_{45}^{-1} & 1 & 0 \\ x_{46}^{-1} & x_{56}^{-1} & 1 \end{pmatrix}$$

For $G_3^3 \simeq B(8,\mathbb{R})$ holds:

$$\begin{pmatrix} 1 & x_{01} & x_{02} & x_{03} & x_{04} & x_{05} & x_{06} & x_{07} \\ 0 & 1 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} \\ 0 & 0 & 1 & x_{23} & x_{24} & x_{25} & x_{26} & x_{27} \\ 0 & 0 & 0 & 1 & x_{34} & x_{35} & x_{36} & x_{37} \\ 0 & 0 & 0 & 0 & 1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} \\ 0 & 0 & 0 & 0 & 0 & 1 & x_{56}^{-1} & x_{57}^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_{67}^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{43} & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{52} & 0 & 0 & 0 & 0 & 0 \\ 0 & y_{61} & 0 & 0 & 0 & 0 & 0 & 0 \\ y_{70} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As before we have

$$B(x,y) = \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} \\ 0 & 1 & x_{56}^{-1} & x_{57}^{-1} \\ 0 & 0 & 1 & x_{67}^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & y_{43} \\ 0 & 0 & y_{52} & 0 \\ 0 & y_{61} & & 0 \\ y_{70} & 0 & & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{01} & x_{02} & x_{03} \\ 0 & 1 & x_{12} & x_{13} \\ 0 & 0 & 1 & x_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{S} = (x(m))^T \, y^T \, (x_m^{-1})^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{45}^{-1} & 1 & 0 & 0 \\ x_{46}^{-1} & x_{56}^{-1} & 1 & 0 \\ x_{47}^{-1} & x_{57}^{-1} & x_{67}^{-1} & 1 \end{pmatrix}$$

New proof of the irreducibility of the induced representations corre-sponding to a generic orbits.

The condition of “maximal possible dimension” is difficult to extend for the infinite-dimensional case. That is why we give another proof of the irreducibility of the induced representation of a nilpotent group $B(n,\mathbb{R})$ that will be extended for the infinite-dimensional analog $B_0^{\mathbb{Z}}$ of the group $B(n,\mathbb{R})$.

Let us consider a sequence of a Lie groups G_n^m and its Lie algebras g_n^m $, m \in \mathbb{Z}, n \in \mathbb{N}$ defined as follows

$$G_n^m = \left\{ I + \sum_{m-n\le k<n\le m+n+1} x_{kn}E_{kn} \right\}, g_n^m = \left\{ \sum_{m-n\le k<n\le m+n+1} x_{kn}E_{kn} \right\}. \qquad (125)$$

We note that for any $m \in \mathbb{N}$ holds $B_0^{\mathbb{Z}} = \lim_{\to n} G_n^m$. We have the decomposition (see(93))

$G_n^m = B_{m,n}B(m,n)B^{(m,n)}$,

where

$$B_{m,n} = \{I + \sum_{(k,r)\in\Delta_{m,n}} x_{kr}E_{kr}\}, B(m,n) = \{I + \sum_{(k,r)\in\Delta(m,n)} x_{kr}E_{kr}\},$$

$$B^{(m,n)} = \{I + \sum_{(k,r)\in\Delta^{(m,n)}} x_{kr}E_{kr}\},$$

and

$$\Delta(m,n) = \{(k,r) \in \mathbb{Z}^2 \mid m-n \le k \le m < r \le m+n+1\},$$
$$\Delta_{m,n} = \{(k,r) \in \mathbb{Z}^2 \mid m+1 \le k < r \le m+n+1\},$$
$$\Delta^{(m,n)} = \{(k,r) \in \mathbb{Z}^2 \mid m-n \le k < r \le m\}.$$

The corresponding elements of the group G_n^m are as follows

$$\begin{pmatrix} 1 & x_{m-n,m-n+1} & \cdots & x_{m-n,m-1} & x_{m-n,m} & t_{m-n,m+1} & t_{m-n,m+2} & \cdots & x_{m-n,m-n+1} \\ 0 & 1 & \cdots & x_{m-n+1,,m-1} & x_{m-n+1,m} & t_{m-n+1,m+1} & t_{m-n+1,m+2} & \cdots & t_{m-n+1,m+n+1} \\ 0 & 0 & \cdots & 1 & x_{m-1,m} & t_{m-1,m+1} & t_{m-1,m+2} & \cdots & t_{m-1,mn+1} \\ 0 & 0 & \cdots & 0 & 1 & t_{m,m+1} & t_{m,m2} & \cdots & t_{m,m+n+1} \\ 0 & 0 & \cdots & 0 & 0 & 1 & x_{m+1,m+2} & \cdots & x_{m+1,m+n+1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & x_{m+2,m+n+1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & x_{m+n,m+n+1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

The induced representation of the group G_n^m is defined in the space $L^2(X,d\mu)$ by the following formula

$$\left(T_t^{m,yn} f\right)(x) = S\big(h(x,t)\big)\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{\frac{1}{2}} f(xt), f \in L^2(X,\mu), x \in X = H\backslash G, t \in G \quad (126)$$

where $X = B(m,n)\backslash G_n^m \cong B_{m,n} \times B^{(m,n)}$ (see (88)),

$$d\mu\big(x_m, x^{(m)}\big) = dx_m \otimes dx^{(m)} = \otimes_{(k,n)\in\Delta m,n}\, dx_{kn} \otimes \otimes_{(k,n)\in\Delta^{(m,n)}}\, dx_{kn} \quad (127)$$

be the Haar measure on the group $B_{m,n} \times B^{(m,n)}$. Denote by $H^{m,n} = L^2(B \times B^{(m,n)}, dx_m \otimes dx^{(m)})$.

Lemma (4.2.14)[161]: Two von Neumann algebra $\mathfrak{A}^S$ and $\mathfrak{A}^x$ in the space $H^{m,n}$ generated respectively by the sets of unitary operators $U_{kr}(t)$ and $V_{kr}(t)$ coincides, where

$$(U_{kr}(t)f)(x) = exp(2\pi i S_{kr}(t))f(x), (V_{kr}(t)f)(x) := exp(2\pi i t x_{kr})f(x),$$

$$\mathfrak{A}^S = \left(U_{kr}(t) = T_{I+tE_{kr}}^{m,yn} = exp(2\pi i S_{kr}(t)) \mid t \in \mathbb{R}, (k,r) \in \Delta(m,n)\right)'',$$

$$\mathfrak{A}^x = \left(V_{kr}(t) := exp(2\pi i t x_{kr}) \mid t \in \mathbb{R}, (k,r) \in \Delta_{m,n} \bigcup \Delta^{(m,n)}\right)''. \quad (128)$$

Proof: Using the decomposition (see (110) and (111))

$$\mathbb{S}_n^{(m)} = (x_m^{-1} y x^{(m)})^T = (x^{(m)})^T\, y^T\, (x_m^{-1})^T \quad (129)$$

we conclude that $\mathfrak{A}^S \subseteq \mathfrak{A}^x$. Indeed, we get $V_{kr}(t) := \exp(2\pi i t x_{kr}) \in \mathfrak{A}^x$ hence the operators x_{kr} of multiplication by the independent variable $f(x) \mapsto x_{kr} f(x)$ in the space $H^{m,n}$ are affiliated with the von Neumann algebra $\mathfrak{A}^x$ $i.e.\, x_{kr}\, \eta\, \mathfrak{A}^x$ for $(k,r) \in \Delta_{m,n} \cup \Delta^{(m,n)}$.

Definition(4.2.15)[161]: Recall (c.f. e.g. [162]) that a non necessarily bounded self-adjoint operator A in a Hilbert space H is said to be affiliated with a von Neumann algebra M of operators in this Hilbert space H, if exp(itA) ∈ M for all $t \in \mathbb{R}$. One then writes $A\, \eta\, M$.

By (115) the matrix elements x_{kr}^{-1} of the matrix $x_m^{-1} \in B_{m,n}$ are also affiliated $x_{kr}^{-1}\, \eta\, Ax$. Using (129) we conclude that the matrix elements $S_{kr}, \in \Delta(m,n)$ of the matrix $\mathbb{S}_n^{(m)}$ are affiliated: $S_{kr}\eta\, \mathfrak{A}^x, (k,r) \in \Delta(m,n)$, so $\mathfrak{A}^S \subseteq \mathfrak{A}^x$.

To show that $\mathfrak{A}^S \supseteq \mathfrak{A}^x$ we find the expressions of the matrix element of the matrix $x^{(m)} \in B^{(m,n)}$ $and\; x_{\;m}^{-1} \in B_{m,n}$ in terms of the matrix elements of the matrix $\mathbb{S}_n^{(m)} = (S_{kr})_{(k,r)\in\Delta(m,n)}$. To do that we connect the above decomposition $\mathbb{S}_n^{(m)} = (x^{(m)})T\, y^T\, (x_m^{-1})^T$ and the Gaussian decomposition C=LDU (see Theorem (4.2.34).

Let us denote by J the n × n anti-diagonal matrix $J = \sum_{r=0}^{n-1} E_{m-r,m+r+1}$ Using $J^2 = I$ and (113) we get

$$\mathbb{S}J = B^T(x,y)J = \left(x^{(m)}\right)^T y^T\, (x_m^{-1})^T J = \left(x^{(m)}\right)^T (y^T J)(J(x_m^{-1})^T J). \quad (130)$$

The latter decomposition (130) is in fact the Gauss decomposition of the matrix $\mathbb{S}J$ i.e.we get

$$\mathbb{S}J = LDU, \text{where } L = (x^{(m)})^T, D = y^T J, U = J(x_m^{-1})^T J.$$

Using the Theorem (4.2.34) we can find the matrix elements of the matrix $x^{(m)} \in B^{(m,n)}$ and $x_m^{-1} \in B_{m,n}$ in terms of the matrix elements of the matrix $\mathbb{S}_n^{(m)}$, hence we can also find the matrix elements of the matrix $x_m \in B_{m,n}$. This finish the proof of the lemma.

We give below the expressions for $\mathbb{S}_n J$. For m = 3 and n = 1 i.e. for G_1^3 we have(remind that $J^2 = I$)

$$\mathbb{S}_2 = \begin{pmatrix} 1 & 0 \\ x_{23} & 1 \end{pmatrix}\begin{pmatrix} 0 & y_{52} \\ y_{43} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_{45}^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x_{23} & 1 \end{pmatrix}\begin{pmatrix} y_{52} & 0 \\ 0 & y_{43} \end{pmatrix}\begin{pmatrix} x_{45}^{-1} & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathbb{S}_2 J = \begin{pmatrix} 1 & 0 \\ x_{23} & 1 \end{pmatrix} \begin{pmatrix} y_{52} & 0 \\ 0 & y_{43} \end{pmatrix} \begin{pmatrix} 1 & x_{45}^{-1} \\ 0 & 1 \end{pmatrix}$$

For G_2^3 we get

$$\mathbb{S}_3 = \begin{pmatrix} 1 & 0 & 0 \\ x_{12} & 1 & 0 \\ x_{13} & x_{23} & 1 \end{pmatrix} \begin{pmatrix} y_{61} & 0 & 0 \\ 0 & y_{52} & 0 \\ 0 & 0 & y_{43} \end{pmatrix} \begin{pmatrix} x_{46}^{-1} & x_{56}^{-1} & 1 \\ x_{45}^{-1} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbb{S}_3 J = \begin{pmatrix} 1 & 0 & 0 \\ x_{12} & 1 & 0 \\ x_{13} & x_{23} & 1 \end{pmatrix} \begin{pmatrix} y_{61} & 0 & 0 \\ 0 & y_{52} & 0 \\ 0 & 0 & y_{43} \end{pmatrix} \begin{pmatrix} 1 & x_{56}^{-1} & x_{46}^{-1} \\ 0 & 1 & x_{45}^{-1} \\ 0 & 0 & 1 \end{pmatrix}.$$

For G_3^3we have

$$\mathbb{S}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{01} & 1 & 0 & 0 \\ x_{02} & x_{12} & 1 & 0 \\ x_{03} & x_{13} & x_{23} & 1 \end{pmatrix} \begin{pmatrix} y_{70} & 0 & 0 & 0 \\ 0 & y_{61} & 0 & 0 \\ 0 & 0 & y_{52} & 0 \\ 0 & 0 & 0 & y_{43} \end{pmatrix} \begin{pmatrix} x_{47}^{-1} & x_{57}^{-1} & x_{67}^{-1} & 1 \\ x_{46}^{-1} & x_{56}^{-1} & 0 & 0 \\ x_{45}^{-1} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbb{S}_4 J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{01} & 1 & 0 & 0 \\ x_{02} & x_{12} & 1 & 0 \\ x_{03} & x_{13} & x_{23} & 1 \end{pmatrix} \begin{pmatrix} y_{70} & 0 & 0 & 0 \\ 0 & y_{61} & 0 & 0 \\ 0 & 0 & y_{52} & 0 \\ 0 & 0 & 0 & y_{43} \end{pmatrix} \begin{pmatrix} 1 & x_{57}^{-1} & x_{67}^{-1} & x_{47}^{-1} \\ 0 & 1 & x_{56}^{-1} & x_{46}^{-1} \\ 0 & 0 & 1 & x_{45}^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}, (131)$$

Theorem(4.2.16)[161]: The induced representation $T^{m,yn}$ of the group G_n^m defined by formula (126), corresponding to generic orbit O_{yn} , generated by the point $yn \in (gmn)*$ $, yn = Pn - 1r = 0\ ym + r + 1, m - rEm + r + 1, m - r$ is irreducible. Moreover the generators of one-parameter groups $A_{kr} = \frac{d}{dt} T^{m,yn}_{I+tE_{kr}}|_{t=0}$ are as follows

$$A_{kr} = \sum_{s=m-n}^{k-1} x_{ks} D_{rs} + D_{kr}, (k,r) \in \Delta^{(m,n)}, A_{kr} = \sum_{s=m+1}^{k-1} x_{ks} D_{rs} + D_{kr}, (k,r) \in \Delta_{m,n},$$

$$(2\pi i)^{-1} (A_{kr})_{(k,r)\in\Delta(m,n)} = S_n^{(m)} = (S_{kr})_{(k,r)\in\Delta(m,n)} = \left(x_m^{-1}\, y x^{(m)}\right)^T.$$

The irreducibility of the induced representation of the group G_n^m is based on the following lemma.

Proof: The irreducibility follows from the Kirillov results . To give another proof of the irreducibility of the induced representation consider the restriction $T^{m,yn}|_{B(m,n)}$ of this representation to the commutative subgroupB(m, n) of the group G_n^m . Note that

$$\mathfrak{A}^x = \left(exp(2\pi i t x_{kr}) \mid t \in \mathbb{R}, (k,r) \in \Delta_{m,n} \bigcup \Delta^{(m,n)}\right)'' = L^\infty(B_{m,n} \times B^{(m,n)}, dx_m \otimes dx^{(m)}).$$

By Lemma(4.2.14) the von Neumann algebra $\mathfrak{A}^S$ generated by this restriction coincides with $L^\infty(B_{m,n} \times B^{(m,n)}, dx_m \otimes dx^{(m)})$. Let now a bounded operator A in a Hilbert space m, ncommute with the representation $T^{m,yn}$. Then A commute by the above arguments with $L^\infty(B_{m,n} \times B^{(m,n)}, dx_m \otimes dx^{(m)})$, therefore the operator A itself is an operator of multiplication by some essentially bounded function $a \in L^\infty$ $i.e. (Af)(x) = a(x)f(x)$ for

$f \in H^{m,n}$. Since A commute with the representation $T^{m,yn}$ i.e. $[A, T_t^{m,yn}] = 0$ for all $t \in B_{m,n} \times B^{(m,n)}$ we conclude that

$$a(x) = a(xt)\ (mod\ dx_m \otimes dx^{(m)})\ for\ all\ t \in B_{m,n} \times B^{(m,n)}.$$

Since the measure $dh = dx_m \otimes dx^{(m)}$ is the Haar measure on $G = B_{m,n} \times B^{(m,n)}$, this measure is G-right ergodic. We conclude that a(x) = const (mod $dx_m \otimes dx^{(m)}$).

Regular and quasiregular representations of infinite-dimensional groups.

To define the induced representation we explain first how to define the regular representation of infinite-dimensional group G. Since the initial group in not locally compact there is neither Haar (invariant) measure on G (Weil, [168]), nor a G-quasi-invariant measure (Xia Dao-Xing, [188]). We can try to find some bigger topological group $\widetilde{G}$ and the G-quasi-invariant measure μ on $\widetilde{G}$ such that G is the dense subgroup in $\widetilde{G}$. In this case we define the right or left regular representation of the group G in the space $L^2(\widetilde{G}, \mu)$ if $\mu^{R_t} \sim \mu$ (resp. $\mu^{L_t} \sim \mu$) for all $t \in G$ as follows:

$$(T_t^{R,\mu} f)(x) = \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{\frac{1}{2}} f(xt), f \in L^2(\tilde{G}, \mu), t \in G, \quad (132)$$

$$(T_t^{R,\mu} f)(x) = (d\mu(t^{-1}x)/d\mu(x))^{1/2} f(t^{-1}x), f \in L^2(\tilde{G}, \mu), t \in G. \quad (133)$$

Conjecture (4.2.17)[161]:(Ismagilov, 1985). The right regular representation $T^{R,\mu} : G \to U(L^2(\widehat{G}, \mu))$is irreducible if and only if

(i) $\mu^{L_t} \perp \mu\ \forall t \in G\backslash\{e\}$,

(ii) the measure μ is G-ergodic.

Analogously we can define the quasiregular representation. Namely, if H is a closed subgroup of the group G, then on the space $X = \widetilde{H\backslash G} = \widetilde{H}\backslash\widetilde{G}$ the right action of the group G is well defined, where $\widetilde{G}$ (resp. ˜H) is some completion of the group G (resp. H). If we have some G-right-quasi-invariant measure μ on X one may define the “quasiregular representation” of the group G in the space $L^2(X, \mu)$ as in a locally compact case:

$$(\pi_t^{R,\mu,X} f)(x) = (d\mu(xt)/d\mu(x))^{1/2} f(xt), t \in G.$$

The regular and quasiregular representations for general infinite-dimensional groups were introduced and investigated in e.g. [123],[9], [142], [143], [146].

The induced representation $\mathrm{Ind}_H^G S$ of a locally-compact group is the unitary representation of the group G associated with a unitary representation S of a subgroup H of the group G as it was mentioned (see [163], [140]) all unitary irreducible representations up to equivalence $\widehat{G}_n$ of the nilpotent group $G_n = B(n, \mathbb{R})$, are obtained as induced representations $\mathrm{Ind}_H^{G_n} U_{f,H}$ associated with a points $f \in g_n^*$ and the corresponding subordinate subgroup $H \subset G_n$. The induced representation $\mathrm{Ind}_H^{G_n} U_{f,H}$ is defined canonically in the Hilbert space $L^2(H\backslash G_{n,\mu})$.

A. Kirillov [140], Chapter I, §4, p.10 says: ”The method of induced representations is not directly applicable to infinite-dimensional groups (or more precisely to a pair G ⊃ H) with an infinite-dimensional factor H\G)”.

We develop the concept of induced representations for infinite-dimensional groups. Let we have the infinite-dimensional group G and a unitary representation $S : H \to U(V)$ in a

Hilbert space V of a subgroup H of the group G such that the factor space H\G is infinite-dimensional.

In general, it is difficult to construct G-quasi-invariant measure on an infinite-dimensional homogeneous space H\G. As is the case of the regular and quasiregular representations of infinite-dimensional groups G it is reasonable to construct some G-quasi-invariant measure on a suitable completion $\widetilde{H\backslash G} = \tilde{H}\backslash\tilde{G}$ of the initial space H\G in a certain topology, where ˜H (resp. $\tilde{G}$) is some completion of the group H (resp.G). To go further we should be able to extend the representation $S : H \to U(V)$ of the group H to the representation $\tilde{S} : \tilde{H} \to U(V)$ of the completion $\tilde{H}$ of the group H.

Finally, the induced representation of the group G associated with a unitary representation S of a subgroup H will depend on two completions ˜H and ˜G of the subgroup H and the group G, on an extension $\tilde{S} : \tilde{H} \to U(V)$of the representation $S : H \to U(V)$ and on a choice of the G-quasi-invariant measure μ on an appropriate completion $\tilde{X} = \tilde{H}\backslash\tilde{G}$ of the space H\G.

Hence the procedure of induction will not be unique but nevertheless well-defined (if a G-quasi-invariant measure on $\widetilde{H\backslash G}$ exists). So the uniquely defined induced representation $\mathrm{ind}_H^G S$ in the Hilbert space $L^2(H\backslash G, V, \mu)$ (in the case of a locally-compact group G) should be replaced by the family of induced representations $\mathrm{Ind}_{\tilde{H},H}^{\tilde{G},G,\mu}(\tilde{S}, S)$ in the Hilbert spaces $L^2(\tilde{H}\backslash\tilde{G}, V, \mu)$ depending on different completions $\tilde{G}$ of the group G, completions ˜H of the group H and different G-quasi-invariant measures μ on $\tilde{H}\backslash\tilde{G}$.

Example(4.2.18)[161]:([141], [143]). Regular representations $T^{R,\mu}$ of the infinite-dimensional group G in the space $L^2(\tilde{G}, \mu)$, associated with the completion ˜G of the group G and a G-right -quasi-invariant measure μ on $\tilde{G}$, is a particular case of the induced representation

$$T^{R,\mu} = \mathrm{Ind}_e^{\tilde{G},G,\mu}(\mathrm{Id}),$$

generated by the trivial representation S = Id of the trivial subgroup H = {e} (as in the case of a locally compact groups).

Example(4.2.19)[161]:([123], [146]). Quasi-regular representations $\pi^{R,\mu,X}$ of the infinite-dimensional group G in the space $L^2(X, \mu)$ where $X = \tilde{H}\backslash\tilde{G}$ and H is some subgroup of the group G is a particular case of the induced representation

$$\pi^{R,\mu,X} = \mathrm{Ind}_{\tilde{H},H}^{\tilde{G},G,\mu}(\mathrm{Id})$$

generated by the trivial representation S = Id of the completion $\tilde{H}$ in the group $\tilde{G}$ of the subgroup H in the group G.

Let G be an infinite-dimensional group and $S : H \to U(V)$ be a unitary representation in a Hilbert space V of the subgroup $H \subset G$, such that the space H\G is infinite dimensional.

We give the following definition.

Definition (4.2.20)[161]: The induced representation

$$\mathrm{Ind}_{\tilde{H},H}^{\tilde{G},G,\mu}(\mathrm{Id})$$

Generated by the unitary representations $S : H \to U(V)$ of the subgroup H in the group G is defined (similarly to (133) and (134)) as follows:

(i) We should first find some completion ˜H of the group H such that

$$\tilde{S}: \tilde{H} \to U(V)$$

Is the continuous unitary representation of the group ˜H , such that $\tilde{S}|_H = S$,

(ii) Take any G-right-quasi-invariant measure μ on the an appropriate completion $\tilde{X} = \tilde{H}\backslash\tilde{G}$ of the space X = H\G, on which the group G acts from the right, where $\tilde{H}$ (resp.$\tilde{G}$) is a suitable completion of the group H (resp. G),

iii) In the space $L^2(\tilde{X}, V, \mu)$ of all vector-valued functions f on ˜X with values in Vsuch that

$$\|f\|^2 := \int_{\tilde{X}} \|f(x)\|_V^2 \, d\mu(x) < \infty,$$

define the representation of the group G by the following formula

$$(T_t f)(x) = S\left(\tilde{h}(x,t)\right)\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{\frac{1}{2}} f(xt), x \in \tilde{X}, t \in G, \qquad (134)$$

where $\tilde{h}$ is defined by

$$\tilde{s}(x)t = \tilde{h}(x,t)\tilde{s}(xt).$$

The section s : H → G of the projection p : G → H should be extended to the appropriate section $\tilde{s}: \tilde{H} \to \tilde{G}$ of the extended projection $\tilde{p}: \tilde{G} \to \tilde{H}$.

The comparison of the induced representation for locally compact group and the above definition for infinite-dimensional groups may be given in the following table:

1	G	Gloc.comp	$DimG = \infty$
2	H	$H \subset G$	$H \subset G$
3	S	$S: H \to U(V)$	$S: H \to U(V) \Rightarrow S: H \to U(V)$
4	X	$X = H / G$	$X = \widetilde{H/G} = \tilde{H} / \tilde{G}$
5	H	$L^2(X = H\backslash G, V, \mu)$	$L^2(X = H\backslash G, V, \mu)$
6	Ind	$Ind_H^G S$	$\mathrm{Ind}_{\tilde{H},H}^{\tilde{G},G,\mu}(\tilde{S}, S)$
7	T_t	$(T_t f)(x) = S(h(x,t))\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1\backslash 2} f(xt)$	$(T_t f)(x) = S(h(x,t))\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1\backslash 2} f(xt)$
8	p	$P: G \to X$	$\tilde{P}: G \to X$
9	s	$s: X \to G$	$S: H\backslash G \to G \Rightarrow \tilde{s}: \widetilde{H\backslash G} \to \tilde{G}$
10	h(x, t)	$s(x)t = h(x,t)s(xt)$	$\tilde{s}(x)t = h(x,t)\tilde{s}(xt)$

Table (1)[161]:

How to develop the orbit method for infinite-dimensional "nilpotent" group $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$? We would like to develop the orbit method for infinite-dimensional"nilpotent" group $G = \lim_{\to n} G_n$ *with* $G_n = B(n, \mathbb{R})$. The corresponding Lie algebra g^* is the inductive *limit* $g = \lim_{\to n} b_n$ of upper triangular matrices, so as the linear space it is isomorphic to the space R_0^∞ of finite sequences $(x_k)_{k\in\mathbb{N}}$ hence the dual space g∗ is isomorphic to the space R^∞ of all sequences $(x_k)_{k\in\mathbb{N}}$ but the latter space R^∞ is too large to manage with it, for example to equip with a Hilbert structure or to describe all orbits.

To make it less it is reasonable to increase the initial group G or to make completion $\tilde{G}$ of this group in some stronger topology.
To develop the orbit method for groups $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$ we should answer some questions:
(a) How to define the appropriate completion $\tilde{G}$ of the group G, corresponding Lie algebras g (resp. $\tilde{g}$) and corresponding dual spaces g^* $(resp. \tilde{g}^*)$?
(b) Which pairing should we use between g and g^*?
(c) Let the dual space g^*, some element $f \in g^*$ and corresponding algebra h, subordinate to the element f, are chosen. How to define the corresponding induced representation $Ind_H^G U_{f,H}$ and study its irreducibility ?
(d) Shall we get all irreducible representations of the corresponding groups, using induced representations?
(e) Find the criteria of irreducibility and equivalence of induced representations.

The problem of completion of the inductive limit group $G = \lim_{\to n} G_n$, where Gn are finite-dimensional classical groups were studied by A. Kirillov ([164], 1972) for the group $U(\infty) = G = \lim_{\to n} U(n)$ and G. Olshanskiĭ([185], 1990) for inductive limit of classical groups. They described all unitary irreducible representations of the corresponding groups $G = \lim_{\to n} G_n$, continuous in stronger topology, namely in the strong operator topology. The description of the dual $\hat{G}$ of the initial group $G = \lim_{\to n} G_n$is much more complicated.

In [165] we have constructed for the group $GL_0(2\infty, \mathbb{R}) = \lim_{\to n} GL(2n-1, \mathbb{R})$ a family of the Hilbert-Lie groups $GL_2(a), a \in \mathfrak{A}$ such that
(a) $GL_0(2\infty, \mathbb{R}) \subset GL_2(a)$ and $GL_0(2\infty, \mathbb{R})$is dense in $GL_2(a)$ for all $a \in \mathfrak{A}$,
(b) $GL_0(2\infty, \mathbb{R}) = \cap_{a\in\mathfrak{A}} GL_2(a)$,
(c) any continuous representation of the group $GL_0(2\infty, \mathbb{R})$ is in fact continuous in some stronger topology, namely in a topology of a suitable Hilbert -Lie group $GL_2(a)$.
(i) Therefore, as we show it is sufficient to consider a Hilbert-Lie completions $B_2(a)$ of the initial group $B_0^{\mathbb{Z}}$.
(ii) In this case the pairing between the corresponding Hilbert-Lie algebra b2(a) and its dual $B_2(a)^*$ is correctly defined by the trace (as in the finite-dimensional case).
(I) We define the induced representations of the group $B_0^{\mathbb{Z}}$ corresponding to a special orbits, generic orbits, using schema given We consider only the simplest example of G−quasi-invariant measures on $\tilde{X} = \tilde{H} \setminus \tilde{G}$, namely the infinite product of one-dimensional Gaussian measures.
(II) How to construct the induced representation corresponding to an arbitrary orbit?

Conjecture(4.2.21)[161]: Two induced representations $\mathrm{Ind}_{H_1}^{G,} \mu_1 U_{f_1,H_1}$ and $\mathrm{Ind}_{\mu_2}^{G,} H_2 U_{f_2,H_2}$ are equivalent if and only if the corresponding measures μ_1 and μ_2 are equivalent and the functionals f_1 and f_2 belong to the same orbit of $(\tilde{g})^*$.
$GL_2(a)$. We show that the Hilbert-Lie groups appear naturally in the representation theory of infinite-dimensional matrix group. The remarkable fact is that for the inductive limit $G = \lim_{\to n} G_n$of matrix groups $G_n \subset GL(2n-1, \mathbb{R})$ it is sufficient to consider only the Hilbert completions of the initial group G and of the spaces $H\backslash G$.

Let us consider the group $GL_0(2\infty,\mathbb{R}) = \lim_{\to n} GL(2n-1,\mathbb{R})$ with respect to the symmetric embedding $i_n^s : G_n \mapsto G_{n+1}, G_n \ni x \mapsto x + E_{-n,-n} + E_{nn} \in G_{n+1}, where\ G_n = GL(2n-1,\mathbb{R})$. We consider here only the real matrices.
The Hilbert-Lie group $GL_2(a)$ we define (see [165]) by its Hilbert-Lie algebra $gl_2(a)$ with composition $[x,y] = xy - yx$

$$gl_2(a) = \{x = \sum_{k,n\in\mathbb{Z}} x_{kn}E_{kn} \mid \|x\|^2_{gl_2(a)} = \sum_{k,n\in\mathbb{Z}} |x_{kn}|^2 a_{kn} < \infty\}, a \in \mathfrak{A}_{GL}\ , GL_2(a)$$

$$= \{I + x \mid (I + x)^{-1} = 1 + y\, x, y \in gl_2(a)\}.$$

To be more precise, let us consider an analogue $\sigma_2(a)$ of the algebra of the Hilbert-Schmidt operators $\sigma_2(H)$ in a Hilbert space H:

$$\sigma_2(a) = \{x = \sum_{k,n\in\mathbb{Z}} x_{kn}E_{kn} \mid \|x\|^2_{\sigma_2(a)} = \sum_{k,n\in\mathbb{Z}} |x_{kn}|^2 a_{kn} < \infty\}.$$

Lemma(4.2.22)[161]: ([165]). The Hilbert space $\sigma_2(a)$is an (associative) Hilbert algebra (i.e. $\|xy\| \le C\|x\|\|y\|, x, y \in \sigma_2(a)$) if and only if the weight $a = (a_{kn})_{(k,n)\in\mathbb{Z}^2}$ belongs to the set $\mathfrak{A}_{GL}$ defined as follows:

$$\mathfrak{A}_{GL} = \{a = (a_{kn})_{(k,n)\in\mathbb{Z}^2} \mid 0 < a_{kn} \le C a_{km} a_{mn}, k, n, m \in \mathbb{Z}, C > 0\}. \quad (135)$$

We define the Hilbert-Lie algebra $gl_2(a)$ as the Hilbert space $\sigma_2(a)$ with an operation $[x,y] = xy - yx$.
Corollary (4.2.23)[161]: The Hilbert space $gl_2(a)$ is a Hilbert-Lie algebra if and only if the weight $a = (a_{kn})_{(k,n)\in\mathbb{Z}^2}$belongs to the set $\mathfrak{A}_{GL}$.
We remark also [179] that $GL_0(2\infty,\mathbb{R}) = \cap_{a\in\mathfrak{A}_{GL}} GL_2(a)$.
Theorem(4.2.24)[161]:(Theorem 6.1 [165]). Every continuous unitary representation U of the group $GL_0(2\infty,\mathbb{R})$ in a Hilbert space H can be extended by continuity to a unitary representation $U_2(a) : GL_2(a) \to U(H)$ of some Hilbert-Lie group $GL_2(a)$ depending on the representation.

Hilbert-Lie groups $B_2(a)$. Let us consider the following Hilbert-Lie group $B_2(a) := B_2^{\mathbb{Z}}(a)$

$$B_2(a) = \{I + x \mid x \in b_2(a)\}, \quad (136)$$

where the corresponding Hilbert-Lie algebra $b_2(a) := b_2^{\mathbb{Z}}(a)$ is defined as

$$b_2(a) = \{x = \sum_{(k,n)\in\mathbb{Z}^2, k<n} x_{kn}E_{kn} \mid x^2_{b_2(a)} = \sum_{(k,n)\in\mathbb{Z}^2, k<n} |x_{kn}|^2 a_{kn} < \infty\}. \quad (137)$$

Lemma(4.2.25)[161]: ([165]). The Hilbert space $b_2(a)$ (with an operation $(x,y) \mapsto xy$) is a Banach algebra if and only if the weight $a = (a_{kn})_{(k,n)\in\mathbb{Z}^2}$ satisfies the conditions

$$a = (a_{kn})_{k<n}, a_{kn} \le C a_{km} a_{mn}, k < m < n, k, m, n \in Z. \quad (138)$$

Denote by $\mathfrak{A}$ the set of all weight a satisfying the mentioned condition.
Take the group $B_0^{\mathbb{Z}}$, fix some its Hilbert completion i.e. a Hilbert-Lie group B2(a), a ∈ A and the corresponding Hilbert-Lie algebra $g = b_2(a)$. The corresponding dual space $g^* = b_2^*(a)$ has the form

$$b_2^*(a) = \{y = \sum_{(k,n)\in\mathbb{Z}^2, k>n} y_{kn}E_{kn} \mid \|y\|^2_{b^2_{b_2^*}(a)} = \sum_{(k,n)\in\mathbb{Z}^2, k>n} |y_{kn}|^2 a_{kn}^{-1} < \infty\}. \quad (139)$$

The adjoint action $B_2(a) \to Aut(b_2(a))$ of the group $B_2(a)$ on its Lie algebra $b_2(a)$ is:

$$b_2(a) \ni x \to Ad_t(x) := txt^{-1} \in b_2(a), t \in B_2(a). \quad (140)$$

The pairing between $g = b_2(a)$ and $g^* = b_2^*(a)$ is correctly defined by the trace:

$$g^* \times g \ni (y,x) \mapsto \langle y,x\rangle := tr(xy) = \sum_{(k,n)\in\mathbb{Z}^2, k>n} x_{kn}y_{nk} \in \mathbb{R}. \quad (141)$$

The coadjoint action of the group $B_2(a)$ on the dual $g^* = b_2^*(a)$ to $g = b_2(a)$ is as follows: for $t \in B_2(x)$ and $y \in b_2^*(a)$

$$t = I + \sum_{(k,n)\in\mathbb{Z}^2, k<n} t_{kn}E_{kn}, y = \sum_{(k,n)\in\mathbb{Z}^2, k<n} t_{kn}E_{kn}, t^{-1} := I + \sum_{(k,n)\in\mathbb{Z}^2, k<n} t_{kn}^{-1}E_{kn}$$

we have

$$(\mathrm{t}^{-1}\mathrm{yt})_{\mathrm{pq}} = \sum_{\mathrm{m}=-\infty}^{q} (\mathrm{t}^{-1}\mathrm{y})_{\mathrm{pm}}\mathrm{t}_{\mathrm{mq}} = \sum_{\mathrm{m}=-\infty}^{q}\sum_{\mathrm{r=p}}^{\infty} \mathrm{t}_{\mathrm{pr}}^{-1}\ \mathrm{y}_{\mathrm{rm}}\mathrm{t}_{\mathrm{mq}}, (\mathrm{p},\mathrm{q}) \in \mathbb{Z}^2, \mathrm{p} > q,$$

hence

$$Ad_x^*(y) = (t^{-1}yt)_- := I + \sum_{\mathrm{m}=-\infty}^{q} (\mathrm{t}^{-1}\mathrm{y})_{\mathrm{pq}}\mathrm{E}_{\mathrm{pq}}\ . \quad (142)$$

We consider four different type of orbits with respect to the coadjoint action of the group $B_2(a)$ in the dual space $b_2^*(a)$.

Case (1) The finite-dimensional orbits corresponding to a finite points $\sum_{(k,n)\in\mathbb{Z}^2, k<n} y_{kn}E_{kn} \in b_2^*(a)$ (finiteness of y means that only finite number of y_{kn} arenonzero). This orbits leads to the induced representations of an appropriate finitedimensional groups $G_n^m, m \in \mathbb{Z}, n \in \mathbb{N}$ defined by (125). All irreducible unitary representations of the groups G_n^m are completely described by the Kirillov orbit method hence the finite-dimensional orbits gives us the set $\bigcup_{n\in\mathbb{N}} \widehat{G_n^m} \subset \widehat{B_0^{\mathbb{Z}}}$ for embedding $\widehat{G_n^m} \subset \widehat{G_{n+1}^m}$).

Case (2) 0-dimensional orbits are of the form:

$$\mathcal{O}_0 = y, y \in b_2^*(a), y = \sum_{k\in\mathbb{Z}} y_{k+1,k}E_{k+1,k}\,.$$

The Lie algebra $b_2(a)$ is subordinate to the functional $y, \langle y, [b_2(a), b_2(a)]\rangle = 0$ since

$$[b_2(a), b_2(a)] = \{x \in b_2(a) \mid x = \sum_{(k,n)\in\mathbb{Z}^2, k<n} x_{kn}E_{kn}\}.$$

The one-dimensional representation of the Lie algebra $b_2(a)$are

$$b_2(a) \ni x \mapsto \langle y, x\rangle = \sum_{k\in\mathbb{Z}} x_{k,k+1}y_{k+1,k} \in \mathbb{R}\,.$$

Corresponding one-dimensional representations of the group $B_2(a)$ are as follows:

$$B_2(a)\ exp(x) \mapsto exp\big(2\pi i(\langle y,x\rangle)\big) = exp\left(2\pi i \sum_{k\in\mathbb{Z}} x_{k,k+1}y_{k+1,k}\right) \in S^1. \quad (143)$$

They are all irreducible and nonequivalent for different $\sum_{k\in\mathbb{Z}} y_{k+1,k}E_{k+1,k} \in b_2^*(a)$.

Case (3) Generic orbit is generated for an arbitrary $m \in \mathbb{Z}$ $by\ a\ point$ $y \in b_2^*(a)$

$$y = \sum_{p=0}^{\infty} y_{m+p+1,m-p}E_{m+p+1,m-p} \in b_2^*(a), with\ y_{m+p+1,m-p} \neq 0, p+1 \in \mathbb{N}. \quad (144)$$

are devoted to the study of this case.
Case (4) General orbits generated by an arbitrary non finite points

$$y = \sum_{(k,n)\in\mathbb{Z},k>n} y_{kn}E_{kn} \in b_2^*(a).$$

Construction of the induced representations of the group $B_0^{\mathbb{Z}}$ corresponding to a generic orbits. Consider more carefully the case (3). The irreducibility we shall study in the following. Take as before the group $B_0^{\mathbb{Z}}$, fix some its Hilbert completion i.e. a Hilbert-Lie group $B_2(a), a \in \mathfrak{A}$, the corresponding Hilbert-Lie algebra $g = b_2(a)$ and its dual $g^* = b_2^*(a)$.
We shall write the analog of the induced representation of the group $B_0^{\mathbb{Z}}$ for generic orbits (see Examples **(**4.2.6) , **(**4.2.7) and**(**4.2.13)) corresponding to the point $y \in b_2^*(a)$ defined by (210) following steps 1)–3) of Definition **(**4.2.20).
Step 1) Extension of the representation $S : H \to U(V)$**.** For fixed $m \in \mathbb{Z}$, consider the decomposition $B^{\mathbb{Z}} = B_m B(m) B^{(m)}$ similar to the decomposition (103), where $B^{\mathbb{Z}} = \{I + \sum_{k,n\in\mathbb{Z},k<n} x_{kn}E_{kn}\}$,

$$B_m = \{I + \sum_{(k,r)\in\Delta_m} m_{xk}rE_{kr}\}, B(m) = \{I + \sum_{(k,r)\in\Delta_m} m_{xk}rE_{kr}\}, B^{(m)}$$
$$= \{I + \sum_{(k,r)\in\Delta_m} m_{xk}rE_{kr}\},.$$

$\Delta_m = \{(k,r) \in \mathbb{Z}^2 \mid m+1 \le k < r\}, \Delta(m) = \{(k,r) \in \mathbb{Z}^2 \mid k \le m < r\}, and\ \Delta^{(m)}$
$= \{(k,r) \in \mathbb{Z}^2 \mid k < r \le m\}$
Since the algebras $h_0(m), m \in \mathbb{Z}$ defined as follows $h_0(m) = \{t - I \mid t \in B_0(m)\}$,where $B_0(m) = B(m) \cap B_0^{\mathbb{Z}}$, are commutative, so $\langle y, [h_0(m), h_0(m)]\rangle = 0$, hence they are subordinate to the functional $y \in g^* = b_2^*(a)$. The corresponding one-dimensional representation of the algebra $h_0(m) = h_0(m) \cap g_0^{\mathbb{Z}}$ is

$$h_0(m) \ni x \longmapsto \langle y, x\rangle = \sum_{p=0}^{\infty} x_{m-p,m+p+1}y_{m+p+1,m-p} \in \mathbb{R}.$$

The unitary representation of the corresponding group $H_0(m)$ is

$$H_0(m) \ni exp(x) \mapsto S(exp(x)) = exp(2\pi i\langle y, x\rangle) \in S^1.$$

This representation can be extended to representation of the corresponding Hilbert-Lie group $H = H_2(m,a) = B(m) \cap B_2(a)$ $(we\ note\ that\ t = exp(t-1))$:

$$H_2(m,a) \ni exp(x) \longmapsto S(exp(x)) = exp(2\pi i\langle y, x\rangle) \in S^1.$$

In what follows we shall use a notation $B_2(m,a)$ for the group $H_2(m,a)$.
Step 2 a) Construction of the completion $\tilde{X} = \tilde{H}\backslash\tilde{G}$ of the space $X = H\backslash G$. It is difficult to construct an appropriate measure on the space $X_{m,0} = B_0(m)\backslash B_0^{\mathbb{Z}}$ since it is isomorphic to the space $R_0^{\infty} \subset R_0^{\infty}$. That is why we consider two homogeneous spaces, an appropriate completions of the space $X_{m,0}$:

$$X_{m,2}(a) = B_{m,2}(a)\backslash B_2(a), X_m = B(m)\backslash B^{\mathbb{Z}}.$$

Since the decompositions holds

$$B_0^{\mathbb{Z}} = B_{m,0}B_0(m)B_0^{(m)}, B_2(a) = B_{m,2}(a)B_2(m,a)B_2^{(m)}(a), B^{\mathbb{Z}} = B_mB(m)B^{(m)},$$

(see Remark(4.2.8)), we have the following inclusions: $X_{m,0} \subset X_{m,2}(a) \subset X_m$, where $X_{m,0} \simeq B_{m,0} \times B_0^{(m)}, X_{m,2}(a) \simeq B_{m,2}(a) \times B_2^{(m)}(a), X_m = B(m)\backslash B^{\mathbb{Z}} \simeq B_m \times B^{(m)}$.
Step 2 b) We construct a measure μ_b on the space X_m with support $X_{m,2}(a)$ i.e. such that $\mu_b(X_{m,2}(a)) = 1$. That is we take $\tilde{X} = \tilde{H}\backslash\tilde{G} = B_2(m,a)\backslash B_2(a)$.
We construct the measure μ_b on the space $X_m \simeq B_m \times B^{(m)}$ as a product-measure $\mu_b = \mu_{b,m} \otimes \mu_b^{(m)}$, where $\mu_{b,m}$ $(resp. \otimes \mu_b^{(m)})$ is Gaussian product measure on the group B_m $(resp. B^{(m)})$ defined as follows:

$$d_{\mu_b,(m)},(x_m) = \otimes_{(k,n)\in\Delta_m d\mu_{b_{kn}}(x_{kn})} = \otimes_{(k,n)\in\Delta_m} \sqrt{\frac{b_{kn}}{\pi}}\, exp(-b_{kn}x_{kn}^2\, dx_{kn}, \qquad (145)$$

$$d_{\mu_b}^{(m)},(x^{(m)}) = \otimes_{(k,n)\in\Delta^{(m)} d\mu_{b_{kn}}(x_{kn})}$$

$$= \otimes_{(k,n)\in\Delta^{(m)} d\mu_{b_{kn}}(x_{kn})} \sqrt{\frac{b_{kn}}{\pi}}\, exp(-b_{kn}x_{kn}^2\, dx_{kn}. \qquad (146)$$

The corresponding Hilbert space is

$$H^m = L^2(X_{m,\mu b}) = L_2(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_b^{(m)}).$$

Lemma(4.2.26)[161]:(Kolmogorov's zero-one law, [159]). We have $\mu_{b,m} \otimes \mu_b^{(m)}(B_{m,2}(a) \times B_2^{(m)}(a)) = 1$ if and only if

$$\sum_{(k,n)\in\Delta(m)\cup\Delta^{(m)}} \frac{a_{kn}}{b_{kn}} < \infty.$$

Lemma(4.2.27)[161]:([141], [142]). The measure $\mu_b = \mu_{b,m} \otimes \mu_b^{(m)}$ is $B_{m,0} \times B_0^{(m)}$ -right-quasi-invariant i.e. $(\mu_b)^{R_t} \sim \mu_b$ for all $t \in B_{m,0} \times B_0^{(m)}$ if and only if

$$S_{kn}^R(\mu b) = \sum_{r=-\infty}^{k-1} \frac{b_{rn}}{b_{rk}} < \infty, for\ all, k < n \leq m.$$

Step 3) The corresponding induced representation of the group $B_0^{\mathbb{Z}}$ we defined as follows:

$$\left(T_t^{m,y} f\right)(x) = S\big(h(x,t)\big)\left(\frac{d\mu_b(xt)}{d\mu_b(x)}\right)^{\frac{1}{2}} f(xt), x \in m_{,m,}\ t \in G, (147)$$

where (see (152))

$$S(h(x,t)) = exp(2\pi i\langle y, h(x,t) - 1\rangle) = exp(2\pi itr\,((t - I)B(x,y))).$$

Consider the induced representation $T^{m,y}$ of the group $B_0^{\mathbb{Z}}$ corresponding to a generic orbit $\mathcal{O}_y$, generated by the point

$$y = \sum_{r=0}^{\infty} y_{m+r+1,m-r} E_{\,m+r+1,m-r} \in b_2^*(a) defined\ by (147). Set\ for\ (k,r) \in \Delta(m)$$

$$S_{kr}(t_{kr}) := \langle\, y, \big(h\big(x, E_{kr}(t_{kr})\big) - I\big)\rangle,$$

then

$$A_{kr} = \frac{d}{dt} exp(2\pi i Skr(t))|_{t=0} = 2\pi i S_{kr}(1). \qquad (148)$$

Let us denote by $\mathbb{s}^{(m)} = \mathbb{s}$ the following matrix (compare with (107) and (108)):

$$\mathbb{s} = (S_{kr})_{(k,r)\in\Delta(m)}, where\ S_{kr} = S_{kr}(1). \quad (149)$$

We calculate now the matrix $\mathbb{S}(t) = (S_{kr}(t_{kr}))_{(k,r)\in\Delta(m)}$ and the matrix $\mathbb{s} = (S_{kr}(1))_{(k,r)\in\Delta(m)}$ using analog of the Lemma **(**4.2.10). As in (106) we have

$\langle y, h(x,t) - I\rangle = tr\,(H(x,t)y) = tr\left(x^{(m)}t_0 x_{m\,y}^{-1}\right) = tr\left(t_0 x_m^{-1} y x^{(m)}\right) = tr\,(t_0 B(x,y)),$

where $t_0 = t - I$ and for $x_m \in B_m, x^{(m)} \in B^{(m)}$ we denote

$$B(x,y) = x_m^{-1} y x^{(m)} \cong \begin{pmatrix}1 & 0\\ 0 & x_m^{-1}\end{pmatrix}\begin{pmatrix}0 & 0\\ y & 0\end{pmatrix}\begin{pmatrix}x^{(m)} & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ x_m^{-1} y x^{(m)} & 0\end{pmatrix}. (150)$$

By definition we have (recall that $E_{kn}(t_{kn}) = I + t_{kn}E_{kn}$)

$$S_{kn}(t_{kn}) = \langle y, (h(x, E_{kn}(t_{kn})) - I)\rangle = tr(t_{kn}E_{kn}B(x,y)),$$

hence by analog of the Lemma **(**4.2.10)we conclude that

$$\mathbb{s} = (S_{kn}(1))_{k,r} = (tr\,(E_{kr}B(x,y)))_{\,k,r} = B^T(x,y) = (x(m))^T\, y^{\,T}(x_m^{-1}\ )^T$$
$$= \begin{pmatrix}0 & (x(m))^T\, y^{\,T}(x_m^{-1}\ )^T\\ 0 & 0\end{pmatrix}. (151)$$

So, we have

$$(S(h(x,t)) = exp(2\pi i\langle y, (h(x,t) - I)\rangle) = exp(2\pi i tr\,((t - I)B(x,y))). (152)$$

Using results of [166] we conclude that the following lemma holds.

Lemma(4.2.28)[161]: The measure $\mu_b = \mu_{b,m} \otimes \mu_b^{(m)}$ is $B_{m,0} \times B_0^{(m)}$ -right-ergodic if

$$E(\mu_b) = \sum_{k<n\le m}\frac{S_{kn}^R(\mu_b)}{b_{kn}} < \infty.$$

Lemma(4.2.29)[161]: Two von Neumann algebra $\mathfrak{A}^S$ and $\mathfrak{A}^x$ in the space $H^m = L^2(X_m, \mu_b)$ generated respectively by the sets of unitary operators $U_{kr}(t)$ and $V_{kr}(t)$coincides, where

$$(U_{kr}(t)f)(x) = exp(2\pi i S_{kr}(t))f(x), (V_{kr}(t)f)(x) := exp(2\pi i t x_{kr})f(x),$$
$$\mathfrak{A}^S = U_{kr}(t) = T_{I+tE_{kr}}^{m,y} = exp(2\pi i S_{kr}(t)) \mid t \in \mathbb{R}, (k,r) \in \Delta(m)'',$$
$$\mathfrak{A}^x = \left(V_{kr}(t) = exp(2\pi i t x_{kr}) \mid t \in \mathbb{R}, (k,r) \in \Delta_m \bigcup \Delta^{(m)}\right)''. \quad (153)$$

Proof. Using the decomposition (151)

$$\mathbb{s}^{(m)} = B(x,y)^T = (x_m^{-1}\, y x^{(m)})^T = (x^{(m)})^T y^T\,(x_m^{-1}\ )^T$$

we conclude that $\mathfrak{A}^s \subseteq \mathfrak{A}^x$ (see the proof of Lemma **(**4.2.14)).

To show that $\mathfrak{A}^s \subseteq \mathfrak{A}^x$ it is sufficient to find the expressions of the matrix element of the matrix $x^{(m)} \in B^{(m)}$ and $x_m^{-1} \in B_m$ in terms of the matrix elements of the matrix $\mathbb{s}^{(m)} = (S_{kr})_{(k,r)\in\Delta(m)}$. To do this we connect the above decomposition $\mathbb{s}^{(m)} = B(x,y)^T$ (see (150)) and the Gauss decomposition $C = LDU$ for infinite matrices (see Theorem**(**4.2.35) . By (150) we get B$(x,y) = x_m^{-1}\ y x^{(m)}$.

To find a matrix connected with the matrix $\mathbb{s}^{(m)}$, for which an appropriate decomposition LDU holds we recall the expressions for $B(x,y)$ for small n and finite-dimensional groups G_n^m (see Example **(**4.2.13)). We note that $J_m^2 = I$, where

$$J_m \in Mat(\infty, \mathbb{R}), J_m = \sum_{r\in\mathbb{Z}} E_{m+r+1,m-r}\,.$$

For G_3^3 we get

$$B(x,y) = x_m^{-1}\, y x^{(m)}$$

$$= \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} \\ 0 & 1 & x_{56}^{-1} & x_{57}^{-1} \\ 0 & 0 & 1 & x_{67}^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & y_{43} \\ 0 & 0 & y_{52} & 0 \\ 0 & y_{61} & 0 & 0 \\ y_{70} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{01} & x_{02} & x_{03} \\ 0 & 1 & x_{12} & x_{13} \\ 0 & 0 & 1 & x_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B(x,y)J = \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} \\ 0 & 1 & x_{56}^{-1} & x_{57}^{-1} \\ 0 & 0 & 1 & x_{67}^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{43} & 0 & 0 & 0 \\ 0 & y_{52} & 0 & 0 \\ 0 & 0 & y_{61} & 0 \\ 0 & 0 & 0 & y_{70} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{23} & 1 & 0 & 0 \\ x_{13} & x_{12} & 1 & 0 \\ x_{03} & x_{02} & x_{01} & 1 \end{pmatrix} \qquad (154)$$

We use the infinite-dimensional analog of the latter presentation, i.e. instead of the group $G_n = B(n,\mathbb{R})$ consider the infinite-dimensional group $B_0^{\mathbb{Z}}$ and do the same. Let

$$xm \in Bm, x(m) \in B(m), y = X\infty r = 0ym+r+1, m-rEm+r+1, m-r \in g*2(a)$$

and $J = J_m = E_{r\in\mathbb{Z}} E_{m+r+1,m-r}$. Then we get $\mathbb{S}^T = B(x,y) = x_m^{-1}\, y x^{(m)}$.

Set $C = C(x,y) = B(x,y)J$ *then* $C = UDL$, more precisely we have:

$$B(x,y)J = x_m^{-1}\, y J_m J_m x^{(m)} J_m = UDL,$$
$$where\ U = x_m^{-1}, D = yJ_m, L = J_m x^{(m)} J_m, \qquad (155)$$

$$C = B(x,y)J$$

$$= \begin{pmatrix} 1 & x_{45}^{-1} & x_{46}^{-1} & x_{47}^{-1} & \cdots \\ 0 & 1 & x_{56}^{-1} & x_{57}^{-1} & \cdots \\ 0 & 0 & 1 & x_{67}^{-1} & \cdots \\ 0 & 0 & 0 & 1 & \cdots \end{pmatrix} \begin{pmatrix} y_{43} & 0 & 0 & 0 & \cdots \\ 0 & y_{52} & 0 & 0 & \cdots \\ 0 & 0 & y_{61} & 0 & \cdots \\ 0 & 0 & 0 & y_{70} & \cdots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ x_{23} & 1 & 0 & 0 & \cdots \\ x_{13} & x_{12} & 1 & 0 & \cdots \\ x_{03} & x_{02} & x_{01} & 1 & \cdots \end{pmatrix}$$

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} & \cdots \\ c_{21} & c_{22} & \cdots & c_{2n} & \cdots \\ & & \cdots & & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} & \cdots \\ & & \cdots & & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & u_{12} & \cdots & c_{1n} & \cdots \\ 0 & 1 & \cdots & c_{2n} & \cdots \\ & & \cdots & & \cdots \\ 0 & 0 & \cdots & c_{nn} & \cdots \\ & & \cdots & & \cdots \end{pmatrix} \begin{pmatrix} d_1 & & \cdots & 0 & \cdots \\ 0 & d_2 & \cdots & 0 & \cdots \\ & & \cdots & & \cdots \\ 0 & 0 & \cdots & d_n & \cdots \\ & & \cdots & & \cdots \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & \\ l_{21} & 1 & \cdots & 0 & \cdots \\ & & \cdots & & \cdots \\ l_{n1} & l_{n2} & \cdots & 1 & \cdots \\ & & \cdots & & \cdots \end{pmatrix} \qquad (156)$$

To finish the proof of the Lemma it is sufficient to find the decomposition (156) $C = UDL$.

Let us suppose that we can find the inverse matrix C^{-1}. Then by (155) holds $C^{-1} = L^{-1}D^{-1}U^{-1}$ and we can use Theorem(4.2.35) to find

$$\mathrm{L^{-1} = J_m(x^{(m)}) - 1J_{(m)}, D^{-1} = y^{-1}J_m, U^{-1} = x_m}.$$

Hence, we can find the matrix elements of the matrix $(x(m))^{-1} \in B^{(m)}$ and $x_m \in B_m$ in terms of the matrix elements of the matrix $C^{-1} = (\mathbb{S}^T J)^{-1} = (B(x,y)J)^{-1}$. Finally, we can also find the matrix elements of the matrix $x^{(m)} \in B^{(m)}$ using formulas (181). This finish the proof of the lemma since in this case we have $x_{kr}\ \eta\mathfrak{A}^s\ for\ (k,r) \in \Delta_m \cup \Delta^{(m)}$. Hence $\mathfrak{A}^s \subseteq \mathfrak{A}^x$.

(i) To find the inverse matrix C^{-1} we write two decompositions:

$$C = L_1 D_1 U_1 = UDL, C^{-1} = (U_1)^{-1}(D_1)^{-1}(L1^{-1}) = L^{-1}D^{-1}U^{-1}. \quad (157)$$

(ii) Using (157) we can find L_1, D_1 and U_1 by Theorem (4.2.35) . More precisely, for all $x \in \Gamma_G$, where

$$\Gamma_C = \{x \in B_m \times B^{(m)} \mid M_{12\ldots k}^{12\ldots k}(C(x)) \neq 0, k \in \mathbb{N}\}$$

holds the decomposition $C(x) = L_1 D_1 U_1$ and the matrix elements of the matrix L_1, D_1 and U_1 are rational functions in $c_{kn}(x)$.

(iii) We can find $(L_1)^{-1}$ and $(U_1)^{-1}$ using formulas (114). Note that $J_m L J_m, U$, and $J_m L^{-1} J_m, U^{-1} \in B_2(a)$.

(iv) Using identity (157) we can calculate $C^{-1} = (U_1)^{-1}(D_1)^{-1}(L_1)^{-1}$, since L^{-1}, D^{-1} and U^{-1} are well defined.

(v) Using equality (157) we can find the decomposition $C^{-1} = L^{-1}D^{-1}U^{-1}$ of the matrix C^{-1} by Theorem (4.2.35)In other words, the decompositions holds $C^{-1} = L^{-1}D^{-1}U^{-1}$ for all $\in \Gamma_{G-1}$, where

$$\Gamma_{C-1} = \{x \in B_m \times B^{(m)} \mid M_{12\ldots k}^{12\ldots k}(C^{-1}(x)) \neq 0, k \in \mathbb{N}\}$$

and the matrix elements of the matrix L^{-1}, D^{-1} and U^{-1} are rational functions in matrix elements $c_{kn}^{-1}(x)$ of the matrixC^{-1}.

Let us denote $(L_1)^{-1} = (L_{1;kn}^{-1})_{kn}, (D_1)^{-1} = diag(d_{1;k}^{-1})k$ and $(U_1)^{-1} = (U_{1;kn}^{-1})_{kn}$. The decompositions $C = L_1 D_1 U_1$ and $C^{-1} = (U_1)^{-1}(D_1)^{-1} \times (L_1)^{-1}$ hold for $x \in \Gamma_C \cap \Gamma_{C-1}$, i.e. almost for all $x \in B_m \times B^{(m)}$ with respect to the measure μ_{b} since $\mu_{\mathrm{b}}(\Gamma_C \cap \Gamma_{C-1}) = 1$. We conclude that the convergence

$$c_{kn}^{-1}(x) = \sum_{m\in\mathbb{N}} U_{1;km}^{-1} d_{1;m}^{-1} L_{1;mn}^{-1}, k, n \in \mathbb{N}$$

holds pointwise almost everywhere $x \in B_m \times B^{(m)} (mod\, \mu_b)$. Since $U_{1;km}^{-1}, d_{1;m}^{-1}$ and $L_{1;mn}^{-1}\, \eta\, \mathfrak{A}^s$ by (ii) and (iii), we conclude by Lemma (4.2.34) that $c_{kn}^{-1}(x)\, \eta\, \mathfrak{A}^s$. This finish the proof of the lemma.

Theorem(4.2.30)[161]: The induced representation $T^{m,y}$ of the group $B_0^{\mathbb{Z}}$ defined by formula (213), corresponding to generic orbit $\mathcal{O}_y$, generated by the point $y = \sum_{r=0}^{\infty} y_{m+r+1,m-r} E_{m+r+1,m-r} \in b_2^*(a)$ is irreducible if the measure $\mu_{b,m} \otimes \mu_b^{(m)}$ on the group $B_m \times B^{(m)}$ is right $B_{m,0} \times B_0^{(m)}$ -ergodic. Moreover the generators of one-parameter groups $A_{kr} = \frac{d}{dt} T_{I+tE_{kr}}^{m,y} |_{t=0}$ are as follows

$A_{kr} = \sum_{s=-\infty}^{k-1} x_{ks} D_{rs} + D_{kr}, (k,r) \in \Delta^{(m)}, A_{kr} = \sum_{s=m+1}^{k-1} x_{ks} D_{rs} + D_{kr}, (k,r) \in \Delta_m$,

$$(2\pi i)^{-1}(A_{kr})_{(k,r)\in\Delta(m)} = \mathbb{s}^{(m)} = (S_{kr})_{(k,r)\in\Delta(m)} = \left(x_m^{-1} y x^{(m)}\right)^T.$$

Here we denote by $D_{kn} = D_{kn}(\mu_b)$ the operator of the partial derivative corresponding to the shift $x \mapsto x + tE_{kn}$ and the measure μ_b on the group $B_m \times B^{(m)} \ni x = I + \sum x_{kr} E_{kr}$:

$$(D_{kn}(\mu_b)f)(x) = \frac{d}{dt}\left(\frac{d\mu_b(x + tE_{kn})}{d\mu_b(x)}\right)^{\frac{1}{2}} f(x + tE_{kn}) |_{t=0}, D_{kn}(\mu_b)$$

$$= \frac{\partial}{\partial x_{kn}} - b_{kn} x_{kn}. \quad (158)$$

The irreducibility of the induced representation of the group $B_0^{\mathbb{Z}}$ follows from the following lemma.

Proof: To show the irreducibility of the induced representation consider the restriction $T^{m,y}\,|_{B_0(m)}$ of this representation to the commutative subgroup $B_0(m)$ of the group $B_0^{\mathbb{Z}}$. Note that

$$\mathfrak{A}^x = \left(exp(2\pi i t x_{kr}) \mid t \in \mathbb{R}, (k,r) \in \Delta_m \bigcup \Delta^{(m)}\right)'' = L^\infty(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_b^{(m)}).$$

By Lemma (4.2.29) the von Neumann algebra $\mathfrak{A}^s$ generated by this restriction coincides with $\mathfrak{A}^x = L^\infty(B_m \times B^{(m)}, \mu_{b,m} \otimes \mu_b^{(m)})$. Let now a bounded operator A in the Hilbert space $\mathcal{H}^m$ commute with the representation $T^{m,y}$. Then A commute by the above arguments with $L^\infty(B_m \times B^{(m)}, \mu \otimes \mu_{b,m} \otimes \mu_b^{(m)})$, therefore the operator A itself is an operator of multiplication by some essentially bounded function $a \in L^\infty\ i.e. (Af)(x) = a(x)f(x)$ for $f \in \mathcal{H}^m$. Since A commute with the representation $T^{m,y}\ i.e. [A, T_t^{m,y}] = 0$ for all $\in B_{m,0} \times B_0^{(m)}$, where $B_{m,0} = B_m \cap B_0^{\mathbb{Z}}\ and\ B_0^{(m)} = B^{(m)} \cap B_0^{\mathbb{Z}}$, we conclude that $a(x) = a(xt)\ (mod\ \mu_{b,m} \otimes \mu_b^{(m)})\ for\ all\ t \in B_{m,0} \times B_0^{(m)}$.

Since the measure $\mu_{b,m} \otimes \mu_b^{(m)}$ on the group $B_m \times B^{(m)}$ is right $B_{m,0} \times B_0^{(m)}$ -ergodic we conclude that $a(x) = const\ (mod\ dx_m \otimes dx^{(m)})$.

First steps. Let $\hat{G}$ be the dual of the group G. Our aim is to describe $\hat{G}$ for $G = \lim_{\rightarrow n} G_n$ where $G_n = B(n, \mathbb{R})$ is the group of all $n \times n$ upper triangular real matrices with units on the principal diagonal, i.e. we would like to describe the dual of the group $B_0^{\mathbb{N}}$ of infinite in one direction and $B_0^{\mathbb{Z}}$ infinite in both directions matrices. Consider the inductive limit $G = \lim_{\rightarrow n} G_n$ of nilpotent groups $G_n = B(n, \mathbb{R})$. The symmetric (resp. nonsymmetric) imbedding gives us two infinite-dimensional analog of "nilpotent" groups $B_0^{\mathbb{Z}}$ (resp. $B_0^{\mathbb{N}}$).

We do not know the description of all $\hat{G}$. We only know that the set $\hat{G}$ contains the following three classes of representations.

(i) The set $\hat{G}$ contains $\cup_n \hat{G}_n\ i.e. \hat{G} \supset \cup_n \hat{G}_n$. One may use Kirillov's orbit method [163], [140] to describe $\hat{G}_n$. The embedding $\hat{G}_n \subset \hat{G}_{n+1}$ is described in Remark (4.2.31).

(ii) We have $\hat{G} \backslash \cup_n \hat{G}_n \neq \emptyset$. Namely $\hat{G} \backslash \cup_n \hat{G}_n$ contains "regular" $T^{R,\mu}$ and "quasiregular" $\pi^{R,\mu,X}$ representations of the group G.

(iii) Induced representations

It is natural together with the group $B_0^{\mathbb{N}}$ (resp. $B_0^{\mathbb{Z}}$) consider all Hilbert-Lie completion $B_2^{\mathbb{N}}\ (a)$ (resp. $B_2^{\mathbb{Z}}\ (a)$) and the group of all upper-triangular matrices $B^{\mathbb{N}}$ (resp. $B^{\mathbb{Z}}$)

$$G_n \to B_0^{\mathbb{N}} \to B_2^{\mathbb{N}}\ (a) \to B^{\mathbb{N}} \to G_n.$$
$$G_n^m \to B_0^{\mathbb{Z}} \to B_2^{\mathbb{Z}}\ (a) \to B^{\mathbb{Z}} \to G_n^m .$$

Together with all imbedding and projections of all mentioned groups $G_n = B(n, \mathbb{R})$ we have:

$$B(n, \mathbb{R}) \overset{i_n^{n+1}}{\rightarrow} B(n+1, \mathbb{R}) \overset{i_n^{\infty}}{\rightarrow} B_0^{\mathbb{N}} \to B_2(a) \to B^{\mathbb{N}} \to B(n+1, \mathbb{R}) \overset{P_{n+1}^n}{\rightarrow} B(n, \mathbb{R}),$$

where the imbedding i_n^{n+1} and the projections P_{n+1}^n are defined as follows:

$$B(n, \mathbb{R}) \ni x \mapsto i_n^{n+1}(x) = x + E_{n+1,n+1} \in B(n+1, \mathbb{R}),$$
$$B(n+1, \mathbb{R}) \ni x = x^{n+1} x_n \mapsto p_{n+1}^n(x) = x_n \in B(n, \mathbb{R}),$$

where $x^{n+1} = I + \sum_{k=1}^n x_{kn+1} E_{kn+1}$, $x_n = I + \sum_{1 \le k < m \le n} x_{km} E_{km}$.

For groups $G_n^m \simeq B(2n,\mathbb{R})$ defined by (125) consider the homomorphism $p_{n+1}^{s,m,n} : G_{n+1}^m \mapsto G_n^m$ defined as follows (for simplicity we define $p_{n+1}^{s,m,n}$ for $m = 0$)

$$G_{n+1}^0 \ni x = x_\uparrow^{n+1} x_n x_\to^n \mapsto p_{n+1}^{s,0,n}(x) = x_n \in G_n^0 ,$$

where

$$x_\uparrow^{n+1} = I + \sum_{-n<k<n+1} x_{k,n+1}E_{k,n+1} , x_\to^n = I + \sum_{-n<k\le n+1} x_{-n,k}E_{-n,k} .$$

Remark(4.3.31)[161]: The embedding $\widehat{B(n,\mathbb{R})} 7 \to \setminus \widehat{B(n+1,\mathbb{R})}$ (resp. $\widehat{G_n^m} \mapsto \widehat{G_{n+1}^m}$) is induced by the homomorphism (125) $p_{n+1}^n : B(n+1,\mathbb{R}) 7 \to B(n,\mathbb{R})$ (resp. by the homomorphism (206) $p_{n+1}^{s,m,n} : G_{n+1}^m \mapsto G_n^m$). So for $m \in \mathbb{Z}$ we get $\bigcup_{n\in\mathbb{N}} \widehat{G_n^{(m)}} \subset \widehat{B_0^{\mathbb{Z}}}$. Similarly, we have $\bigcup_{n\in\mathbb{N}} \widehat{B(n,\mathbb{N})} \subset B_0^{\mathbb{N}}$

Let us denote by $B_2^{\mathbb{N}}(a)$ (resp. $B_2^{\mathbb{Z}}(a)$) the completion of the subgroup $B_0^{\mathbb{N}} \subset GL_0(2\infty,\mathbb{R}) (resp. B_0^{\mathbb{Z}} \subset GL_0(2\infty,\mathbb{R}))$ in the Hilbert-Lie group $GL_2(a)$. Since (see [165])

$$B_0^{\mathbb{N}} = \bigcap_{a\in\mathfrak{A}} B_2^{\mathbb{N}}(a) \ (resp. B_0^{\mathbb{Z}} = \bigcap_{a\in\mathfrak{A}} B_2^{\mathbb{Z}}(a))$$

we conclude that

$$\widehat{B_0^{\mathbb{N}}} = \bigcup_{a\in\mathfrak{A}} \widehat{B_2^{\mathbb{N}}(a)} \ (resp. \widehat{B_0^{\mathbb{Z}}} = \bigcup_{a\in\mathfrak{A}} \widehat{B_2^{\mathbb{Z}}(a)}).$$

It leaves to describe $\widehat{B_2^{\mathbb{N}}(a)}$ (resp. $\widehat{B_2^{\mathbb{Z}}(a)}$) for all $a \in \mathfrak{A}$. The problem of developing the orbit method for the Hilbert-Lie group $B_2^{\mathbb{N}}(a)$ (resp. $B_2^{\mathbb{Z}}(a)$) could be easier, since the corresponding Lie algebra $b_2^{\mathbb{N}}(a)$ (resp. $b_2^{\mathbb{Z}}(a)$) is a Hilbert-Lie algebra, the dual $(b_2^{\mathbb{N}}(a))^*$ (resp. $(b_2^{\mathbb{Z}}(a))^*$) and the pairing between $B_2^{\mathbb{Z}}(a)$ (resp. $B_2^{\mathbb{Z}}(a)$) and $(b_2^{\mathbb{N}}(a))^*$ (resp. $(b_2^{\mathbb{Z}}(a))^*$) are well defined

Using (206) we conclude

$$B_0^{\mathbb{Z}} = \varinjlim_{n,i} B(n,\mathbb{R}) , B_0^{\mathbb{N}} = \varprojlim_{a} B_2^{\mathbb{N}}(a), B^{\mathbb{N}} = \lim_{n,p} B(n,\mathbb{R}) ,$$

$$\widehat{B_0^{\mathbb{N}}} \supset \widehat{B_2^{\mathbb{N}}(a)} \supset \widehat{B^{\mathbb{N}}} \tag{159}$$

finally we conclude that

$$\widehat{B_0^{\mathbb{N}}} = \bigcup_{a\in\mathfrak{A}} \widehat{B_2^{\mathbb{N}}(a)}, \qquad \widehat{B^{\mathbb{N}}} = \bigcup_{n\in\mathbb{N}} \widehat{G_n} = \bigcup_{n\in\mathbb{N}} \widehat{B(n,\mathbb{R})}. \tag{160}$$

The similar relations holds also for groups $B_0^{\mathbb{Z}} \subset B_2^{\mathbb{Z}}(a) \subset B^{\mathbb{Z}}$.

Definition(4.2.32)[161]: We call the representation of the group $G = \lim_{\to n} G_n$ local if it depends only on the elements of the subgroup G_n for some fixed $n \in \mathbb{N}$.

The last relation in (159) and (160) we can reformulated as follows:

Theorem(4.2.33)[161]: (V.L. Ostrovsky, PhD dissertation, 1986). The class of all irreducible unitary local representations of the group $B_0^{\mathbb{N}} = \lim_{\to n} B(n,\mathbb{R})$ coincides with the class $\bigcup_n \hat{G}_n$.

Gauss decomposition of $n \times n$ matrices. We need some decomposition of the matrix $C \in Mat(n,\mathbb{C})$. Let us denote by

$$M_{j_1 j_2 \dots j_r}^{i_1 i_2 \dots i_r}(C), 1 \le i_1 < \dots < i_r \le n, 1 \le j_1 < \dots < j_r \le n$$

the minors of the matrix C with $i_1, i_2, \dots, i_r$ rows and $j_1, j_2, \dots, j_r$ columns.

Theorem (4.2.34)[161]:(Gauss decomposition, [133]). A matrix $C \in \mathrm{Mat}(n,\mathbb{C})$ admits the following decomposition $C = LDU$ (Gauss decomposition),

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ & & \cdots & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ & & \cdots & \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & d_n \end{pmatrix} \begin{pmatrix} 1 & & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ & & \cdots & \\ 0 & 0 & \cdots & 1 \end{pmatrix} \quad (161)$$

where L (resp. U) is lower (resp. upper) triangular matrix and D a diagonal matrix if and only if all principal minors of the matrix C are different from zeros i.e. $M^{1,2,\dots,k}_{1,2,\dots,k}(C) \neq 0, 1 \leq k \leq n$. Moreover the matrix elements of the matrices L, U and D are given by the formulas (see [133])

$$l_{mk} = \frac{M^{1,2,\dots,k-1,m}_{1,2,\dots,k-1,k}(C)}{M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(C)}, u_{km} = \frac{M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,m}(C)}{M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(C)}, 1 \leq k < m \leq n, \quad (162)$$

$$d_1 = M^1_1(C), d_k = \frac{M^{1,2,\dots,k}_{1,2,\dots,k}(C)}{M^{1,2,\dots,k-1}_{1,2,\dots,k-1}(C)}, 2 \leq k \leq n, \quad (163)$$

Proof: If we write $L^{-1}C = DU$, we get

$$M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(C) = M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(L^{-1}C) = M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,k}(DU) = d_1 \dots d_k,$$

this implies (163). Moreover, we get also

$$M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,m}(L^{-1}C) = M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,m}(C) = M^{1,2,\dots,k-1,k}_{1,2,\dots,k-1,m}(DU) = d_1 \dots d_k u_{km}, k < m,$$

this implies the second formula in (162). Similarly if we write $CU^{-1} = LD$ we get

$$M^{1,2,\dots,k-1,m}_{1,2,\dots,k-1,k}(CU^{-1}) = M^{1,2,\dots,k-1,m}_{1,2,\dots,k-1,k}(C) = M^{1,2,\dots,k-1,m}_{1,2,\dots,k-1,k}(LD) = d_1 \dots d_k l_{mk}, k < m,$$

this implies the first formula in (162).

Let us consider the infinite matrix $C, L, D, U \in \mathrm{Mat}(\infty,\mathbb{C})$.

Theorem(4.2.35)[161]:(Gauss decomposition $C = LDU$). A matrix $C \in \mathrm{Mat}(\infty,\mathbb{C})$ admits the following decomposition $C = LDU$ (Gauss decomposition),

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} & \cdots \\ c_{21} & c_{22} & \cdots & c_{2n} & \cdots \\ & & \cdots & & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} & \cdots \\ & & \cdots & & \cdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \\ l_{21} & 1 & \cdots & 0 & \\ & & \cdots & & \cdots \\ l_{n1} & l_{n2} & \cdots & 1 & \cdots \\ & & \cdots & & \cdots \end{pmatrix} \begin{pmatrix} d_1 & 0 & \cdots & 0 & \cdots \\ 0 & d_2 & \cdots & 0 & \cdots \\ & & \cdots & & \cdots \\ 0 & 0 & \cdots & d_n & \cdots \\ & & \cdots & & \cdots \end{pmatrix} \begin{pmatrix} 1 & u_{12} & \cdots & u_{1n} & \cdots \\ 0 & 1 & \cdots & u_{2n} & \cdots \\ & & \cdots & & \cdots \\ 0 & 0 & \cdots & 1 & \cdots \\ & & \cdots & & \cdots \end{pmatrix} \quad (164)$$

where L (resp. U) is lower (resp. upper) triangular matrix and D a diagonal matrix of infinite order if and only if all principal minors of the matrix C are different from zeros i.e.

$M_{1,2,\dots,k}^{1,2,\dots,k}\ (C)\ \neq, k\ \in\ \mathbb{N}$. Moreover the matrix elements of the matrices L, U and D are given by the same formulas as in the Theorem (4.2.34):

$$l_{mk}\ =\ \frac{M_{1,2,\dots,k-1,k}^{1,2,\dots,k-1,m}\ (C)}{M_{1,2,\dots,k-1,k}^{1,2,\dots,k-1,k}\ (C)}\ , u_{km}\ =\ \frac{M_{1,2,\dots,k-1,m}^{1,2,\dots,k-1,k}(C)}{M_{1,2,\dots,k-1,k}^{1,2,\dots,k-1,k}\ (C)}\ , k, m\ \in\ \mathbb{N}, k\ <\ m, \qquad (165)$$

$$d_1\ =\ M_1^1(C), d_k\ =\ \frac{M_{1,2,\dots,k}^{1,2,\dots,k}\ (C)}{M_{1,2,\dots,k-1}^{1,2,\dots,k-1}\ (C)}\ , k\ \in \mathbb{N}, k\ >\ 1. \qquad (166)$$

Proof: The proof repeat word by word the proof of the Theorem(4.2.34) .

Let $(X, \mathcal{F}, \mu)$ be a measurable space, with a finte measure $\mu(X)\ <\ \infty$, where $\mathcal{F}$ is a sigma-algebra. Consider the set $(f_n)\ =\ (f_n)_{n\in\mathbb{N}}$ of measurable real valued functions on X i.e. $f_n:\ X\ \longmapsto \mathbb{R}$. Denote by $B(H)$ the von Neumann algebra of all bounded operators in the Hilbert space $H\ =\ L^2(X, \mu)$ and let $\mathfrak{A}^{(fn)}(\in\ B(H))$ be a von Neumann algebra generated by operators $U_n(t)$ of multiplication by functions $\exp(itf_n(x)), n\ \in \mathbb{N}$

$$\mathfrak{A}^{(f_n)} =\ \left(U_n(t)\ =\ e^{itf_n}\mid n\ \in \mathbb{N}, t\ \in\ \mathbb{R}\right)''.$$

We are interesting in the following question. Let $f_n\ \to\ f\ as\ n\ \to\ \infty$ in some sense.

When $U(t)\ =\ e^{itf}\ \in\ \mathfrak{A}^{(f_n)}\ for\ all\ t\ \in \mathbb{R}$?

Since $\mathfrak{A}^{(f_n)}$ is a von Neumann algebra it is sufficient to find when the strong convergence of the unitary operators in the space H holds i.e. s. $\lim_n U_n(t)\ =\ U(t)$, where the operators $U_n(t), n\ \in \mathbb{N}$ and $U(t)$ are defined as follows

$$(U_n(t)g)(x)\ =\ e^{itf_n(x)}g(x), (U(t)g)(x)\ =\ e^{itf(x)}\ g(x), g\ \in\ L^2(X, \mu), t\ \in \mathbb{R}.$$

Lemma (4.2.36)[161]: Let $f_n\ \to\ f\ as\ n\ \to\ \infty$ pointwise almost everywhere, then s . $\lim_n U_n(t)\ =\ U(t)\ hence\ U(t)\ =\ e^{itf}\ \in\ \mathfrak{A}^{(f_n)}$.

Proof: For $g\ \in\ H$ we get

$$\|(U_n(t)\ -\ U(t))g\|^2\ = \int_X |\left(e^{itf_n(x)}\ -\ e^{itf(x)}\right)g(x)\mid{}^2 d\mu(x)$$

$$= \int_X \mid e^{itf_n(x)-itf(x)}\ -\ 1\mid^2\mid g(x)\mid^2 2d\mu(x)$$

$$= \int_X \mid e^{it\alpha_n(x)}\ -\ 1\mid^2\mid g(x)\mid^2\ d\mu(x)\ \to\ 0$$

as $n\ \to\ \infty$, if $\alpha_n(x) :=\ f_n(x)\ -\ f(x)\ \to\ 0$ pointwise almost everywhere by Lebesgue's dominated convergence theorem.

Corollary (4.2.37)[260]: Let $f_n^m\ \to\ f^m$ as $n\ \to\ \infty$ pointwise almost everywhere, then s. $\lim_n U_n^m(t)\ =\ U^m(t)$ hence $\sum_m\ U^m(t)\ = \sum_m\ e^{itf^m}\ \in\ \mathfrak{A}^{(f_n^m)}$.

Proof: For $g\ \in\ H$ we get

$$\sum_m \|(U_n^m(t) - U^m(t))g\|^2 = \int_X \sum_m |\left(e^{itf_n^m(x)} - e^{itf^m(x)}\right)g(x)\,|^2 d\mu(x)$$

$$= \int_X \sum_m | e^{itf_n^m(x) - itf^m(x)} - 1 |^2 | g(x) |^2 2 d\mu(x)$$

$$= \int_X \sum_m | e^{it\alpha_n(x)} - 1 |^2 | g(x) |^2 d\mu(x) \to 0$$

as $n \to \infty$, if $\alpha_n(x) := f_n^m(x) - f^m(x) \to 0$ pointwise almost everywhere by Lebesgue's dominated convergence theorem.

Chapter 5
Some Problems and Baire Measurability with Borel Structures

We concern the Borel structures in $C(K)$ generated by the norm, weak or pointwise topology in $C(K)$. We give an example of a compact space K such that the weak and the pointwise topology generate different Borel structures in $C(K)$. We discuss the coincidence of the Baire σ-algebras on $C(K)$ associated to the weak and point wise convergence topologies. ($\beta\omega$). We show that the Borel structures in C ($\beta\omega$) generated by the weak and the pointwise topology are also different. We also show that in C (ω^*) where $\omega^* = \beta\omega/\omega$there is no countable family of pointwise Borel sets separating functions from C (ω^*).

Section (5.1): Borel Structures in Function Space

Given a space C(K) of continuous real-valued functions on a compact space K, we shall consider thefollowing four σ-algebras in $C(K)$: the cylindrical σ-algebra Cyl(C(K)), i.e., the smallest σ-algebra, for which all functionals from the dual space $C(K)^*$ are measurable, cf. [177], [193], and the σ-algebras Borel ($C(K)$, norm), Borel(C(K), weak), Borel(C(K), pointwise) of Borel sets in $C(K)$ with respectto the uniform topology, the weak topology, or the pointwise topology in C(K), respectively.

We shall discuss two topics. The first one concerns the problem which separable compact spaces Khave the property that the measurable space (C(K), Cyl(C(K))) is a standard Borel space, i.e., there is abijection of C(K) onto the irrationals ω^ω taking the elements of Cyl(C(K)) to Borel sets in ω^ω and viceversa.

There is a conjecture that, among separable compact spaces K, this property of $C(K)$ characterizes exactly the compacta which can be embedded in the space $B_1(\omega^\omega)$ of the first Baire class functions on the Cantor set, equipped with the pointwise topology.

Using recent results of Dodos [172] one can show that, indeed, for all separable compacta$K \subset B_1(2^\omega)$,the measurable space (C(K), Cyl(C(K))) is standard, cf. [185].

As showd in [185], the conjecture is true for separable compact spaces whose subspace of accumulation points has exactly one non-isolated point.

We confirm also the conjecture in the class of separable linearly ordered compact spaces.

The second topic concerns the relations between the Borel structures in C(K) generatedby the uniform topology, the weak topology or the topology of pointwise convergence. This subject was originated in Edgar [174], [175]. Talagrand [69], answering a question from [174], showd that $Borel(C(\beta\omega), norm) \neq Borel(C(\beta\omega),$weak). We shall show that if C(K) is a function space representation of the algebra L^∞determined by the Lebesgue measure λ on [0, 1], cf. [171], [191] (anequivalent description of K is that K is the Stone space of the measure algebra associated with λ, cf. [179]), then the inclusions between any of the three Borel structures inC(K) are strict.

We did not find similar results, cf. [190], [177]. It is also not clear to us if $C(\beta\omega)$ has this property.

We shall denote by $B_1(M)$ the space of real-valued first Baire class functions on a separable metrizable space M, equipped with the topology of pointwise convergence. Rosenthal compacta are compact spaces which can be embedded in $B_1(\omega^\omega)$, cf [178]. Let us recallan important characterization of separable Rosenthal compacta due to Godefroy, introducing first somenotation.

Let K be a separable compact space. For each countable set D dense in K, we consider

$$C_D(K) = \{f \mid D : f \in C(K)\} \subset \mathbb{R}^D, \quad (1)$$

i.e., the set of restrictions of continuous functions on $K$to D, which is a subspace of the countable productof the real line. The space $C_D(K)$ can be identified with the topological space $(C(K), \mathcal{T}_D)$, where $\mathcal{T}_D$ isthe topology of pointwise convergence on D.

Now, a separable compact space K is a Rosenthal compactum if, and only if, for any countable set Ddense in K, the set C_D (K) is analytic, cf. [178].

We should mention that there are separable compact subspaces of $B_1(\omega^\omega)$, which do not embed in $B_1(2^\omega)$,, cf. [189]. There are even linearly orderable spaces with such properties, see Example **(**5.1.6).

Following [178], let us check that for a separable compact space K, the measurable space (C(K),Cyl(C(K))) is standard if, and only if, for each countable set D dense in K, C_D (K) is a Borel set in $\mathbb{R}^D$,cf. [178].

Indeed, if D is a countable, dense set in K, under the restriction map $C(K) \to \mathbb{R}^D$ the preimagesof Borel sets belong to $Cyl(C(K))$, hence if $(C(K), Cyl(C(K)))$ is standard, so is the measurable space($C_D(K)$, Borel($C_D(K)$,))), cf. (1), and therefore, ($C_D(K)$, is a Borel set in $\mathbb{R}^D$, cf. [82].

Conversely, if for each countable set D dense in K, $C_D(K)$ is Borel in $\mathbb{R}^D$, $K$is a Rosenthal compactumby the Godefroy theorem, hence a Frechet space, and therefore the functionals from $C(K)^*$areBorel functions on $(C(K),),_{\mathcal{T}D})$ cf. [178]. In effect, the identity

$$(C_D(K), Borel(C_D(K))) \to (C(K), Cyl(C(K)))$$

is an isomorphism of the measurable spaces, and hence the space $(C(K), Cyl(C(K)))$ is standard.

It is not clear if a compact space K must be separable, whenever the space $(C(K), Cyl(C(K)))$ isstandard.

Having explained background, let us reiterate the open part of the conjecture.

Problem (5.1.1)[170]: Let K be a separable compact space such that the function space equipped with the cylindricalσ-algebra $(C(K), Cyl(C(K)))$ is a standard measurable space. Does the compactum K embed in $B_1(2^\omega)$?

Proposition (5.1.2)[170]: Let M be a Borel subspace of a separable completely metrizable space, and let K be a compact subspace of $B_1(M)$ such that, for every $f \in K$the set of discontinuity points of f is countable. Then K can be embedded into the subspace of $B_1(2^\omega$ consisting of functions with only countably manydiscontinuities.

Proof: Let M_1 be the set of all condensation points of the space M. We put$M_2 = M \setminus M_1$ and we consider the projections $p_i\ of\ \mathbb{R}^M\ on\ to\ \mathbb{R}^{M_i}$for $i = 1, 2$. The set M_2 is countable, therefore the projection$p_2(K)$ is metrizable. Since the Banach space $C(2^\omega)$ is universal for the class of all separable metrizablespaces (cf. [45]) and the pointwise topology is weaker than the norm topology, we can find anembedding $h_2\ of\ p_2(K)$ into $C_p(2^\omega). If\ M_1 = \emptyset$then we are done. In the opposite case we proceed withthe construction of the required embedding as follows.

Fix a metric ρ in 2^ω. Let Q be a countable dense subset of 2^ω. Then the complement $P = 2^\omega \setminus Q$ishomeomorphic to the space of the irrationals ω^ω, and therefore we can find a continuous injective map φ of ρ onto M_1 , see [181].We enumerate Q as $\{ q_n : n \in \mathbb{N}\}$and we take a sequence$(p_n)_{n\in\mathbb{N}}$of distinct points of P such that $\rho(p_n, q_n) < 1/n$for n∈ ℕ. For each $f \in K$define the function $gf : 2^{\omega} \to \mathbb{R}$by

$$gf(x) = \begin{cases} f(\varphi(x)) & if \quad x \in P \\ f(\varphi(P_n)) & if \quad x \in q_n \end{cases}$$

For$x \in 2^{\omega}$· Denote by N_f the set of all points of discontinuity of f. One can easily verify that the functiong_f is continuous at every point of the set $P\backslash\varphi^{-1}(N_f)$. Hence the set of points of discontinuity of g_f iscountable and it follows that g_f is of the first Baire class, see [181].

Finally, put $C = 2^{\omega} \times \{1, 2\}$and define $h: K \to B_1(C)$ as follows:

$$h(f)((x,i)) = \begin{cases} gf(x) & if \quad i = 1 \\ h_2(P_2(f))(x) & if \quad i = 2 \end{cases}$$

for $(x, i) \in C$. Clearly, C is a topological copy of the Cantor set, and a routine verification shows that hisan embedding.

We shall use the following well-known description of the class of separable compact linearly ordered spaces.

Let A be an arbitrary subset of a closed subset K of the unit interval [0, 1]. Put

$$K_A = (K \times \{0\}) \cup (A \times \{1\})$$

and equip this set with the order topology given by the lexicographical order $(i.e., (s,i) \prec (t,j)$ if either $s < t$, or $s = t$and $i < j)$. Ostaszewski [209] showd that every separable compact linearly ordered space is homeomorphic to K_A for some closed set $K \subset [0,1]$ and a subset $A \subset K$.

Proposition(5.1.3)[170]: Let L be a separable compact linearly ordered space. If, for some countable dense$D \subset L$, the space $C_D(L)$ is Borel, then L embeds into $B_1(2^{\omega})$.

Proof: By the mentioned above result of Ostaszewski we can assume that $L = K_A$ for a certain closed set K in [0, 1] and a subset A ⊂ K. By [183] there exists a countable dense subset $E \subset K_A$containing D, and a countable C ⊂A such that the space $C_E(K_A)$ contains a closed copy of the set A \ C. Since K_A is a first countable space, the identity map between $C_D(K_A)$ and $C_E(K_A)$ (taking, for any $f \in C(K)$, the restriction $f|D \; to \; f|E$) is a Borel isomorphism, see the proof of Theorem 2.2 in [182]. Therefore, the space $C_E(K_A)$is Borel, and this implies that $A \setminus C$ and $A$are Borel subsets of [0, 1]. Put $M = A \cup \{2\}$. We define a map $\varphi: K_A \to B_1(M)$ by

$$\varphi((t,i))(s) \begin{cases} t & if \; s = 2, \\ 0 & if \; s \in A, \; and \; t > s, \\ 1 & if \; s \in A, and \; t < s, \\ 0 & if \; t \in A, \; i = 1, and \; t = s, \\ 1 & if \; t \in A, i = 0, and t = s, \end{cases}$$

for$(t,i) \in K_A$ and $s \in M$. A routine verification shows that $\varphi$is continuous and injective, hence it is anembedding. For every $(t,i) \in K_A$, the function $\varphi((t,i))(\cdot)$ has at most one discontinuity point (possibly t, when $t \in A$). Therefore, from Proposition (5.1.2) it follows that K_A embeds into $B_1(2^{\omega})$.

Obviously, Proposition (5.1.3) implies the implication (ii) ⇒ (i) in the theorem below. As was mentioned, the reverse implication, for all separable compact spaces, can be found in [185].

Theorem(5.1.4)[170]: For a separable compact linearly ordered space L the following conditions are equivalent:

(i) L embeds in $B_1(2^\omega)$,

(ii) the space $C_D(L)$ is Borel for every countable dense $D \subset L$.

Using Theorem (5.1.4) and [183] one can easily verify the properties of the following example.

Example(5.1.5)[170]: Let A be an analytic non-Borel subset of the unit interval $I = [0, 1]$. Then the space I_A is a separable compact linearly ordered space which is a Rosenthal compactum, but does not embed in $B_1(2^\omega)$.

Theorem (5.1.4), if $B_1(2^\omega)$. is linearly ordered, or by a result from [185], in the other case. Therefore, the countable product K also embeds in $B_1(2^\omega)$.

We shall show (varying an approach used by Dennis Burke and in [72]) that for the Lebesgue measure λ on [0, 1], the set $\{ f \in L^\infty : \int f d\lambda > 0 \}$is not Borel in the topology generated on L^∞ by the multiplicative functionals in the dual space (L^∞). In particular, representing the algebra L^∞ as a function space $C(K)$, we have $Borel(C(K), weak) \neq Borel(C(K)$,pointwise). Since $L^\infty$is linearly isomorphic to $C(\beta\omega)$, cf. [186], by the result of Talagrand mentioned, we obtain the following example.

Example(5.1.6)[170]: There exists a compact space Ksuch that

$$Borel(C(K), norm) \neq Borel(C(K), weak \neq Borel(C(K), pointwise).$$

Problem (5.1.7)[170]: Let $Borel(C(K), norm) = Borel(C(K), weak)$. Is it true that also Borel(C(K), weak) = Borel(C (K), pointwise)?

If *C (K)* admits a Kadec (pointwise-Kadec) norm then $Borel(C(K),$ norm$)$ = Borel($C(K)$, weak)(Borel(C(K), norm) =Borel(C(K),pointwise)). We do not know of any instances where C(K) admits a Kadec norm, but fails to admit a pointwise-Kadec norm, cf. [190].

Let us notice that in some models of set theory, there are compact spaces K such that

Borel(C (K), norm) = Borel(C (K), weak) = Borel(C(K), pointwise),

But $C(K)$ admits no Kadec norm; see [184]. We do not know any such spaces constructed in ZFC.

The above mentioned space K considered in [184] has in addition the property that there is a countable dense set D in K such that the restriction map $C(K) \rightarrow C_D(K)$, cf. (1), takes norm-Borel sets in C(K) to Borel sets in $C_D(K)$ and vice versa. In particular, Borel(C (K), norm)=Cyl(C(K)), cf. [176]. This can be verified by noticing that for the space K discussed in [184], the pointwise Borel sets considered in [184], belong in fact to the cylindrical σ-algebra in C(K).

We shall now pass to a proof of the property of the space L^∞stated at the beginning. We shall obtain some stronger results, in a more general setting. Following Oxtoby [188] we shall call a nonnegative Radon measure μ on a compact space K a category measure, if μ-null sets coincide with meagre sets in K, cf. also [179].

If ν is any non-atomic probability measure and *C (K)* is a function space representation of the algebra $L^{\infty}(\nu)$, then $K$is an extremally disconnected compact space and ν gives rise to a non-atomic probability category measure on K, cf. [180].

Let us recall that in a topological space X, the elements of the smallest σ-algebra in X containing open sets and closed under the Souslin operation are called C-sets, cf. [82]. The C-sets are open modulo meager sets and any preimage of a C-set under a continuous map is a C-set.

Let us also recall that a Radon measure ν on a compact space $K$is singular with respect to a nonnegative Radon measure μon Kif $\nu$is concentrated on a μ-null set.

Theorem (5.1.8)[170]: Let μbe a non-atomic probability category measure on an extremally disconnected compact space K. Then the set $\{ f \in C(K) : \int f d\mu > 0 \}$is not a C-set in the topology generated on *C(K)* by the Radon measures singular with respect to μ.

Proof: Since μ is a category measure, the closure of a μ-null set in K is μ-null, cf. [188], [179]. In particular, a Radon measure on K is singular with respect to μ if and only if its support is a μ-null set.

Let Z be the collection of closed boundary sets in K, i.e., the collection of closed μ-null sets in K. We shall denote by T_Z the topology in $C(K)$ whose basic open sets are defined by

$$\mathcal{N}(f, \mathcal{Z}) = \{ g \in C(K) : f|\mathcal{Z} = g|\mathcal{Z} \}, \qquad \mathcal{Z} \in \mathcal{Z}.$$

The topology $\mathcal{T}_Z$ is stronger than the topology in $C(K)$ generated by the Radon measures singular with respect to μ. Therefore it is enough to show that

$$\{ f \in C(K) : \int f d\mu > 0 \} \ is\ not\ a\ C - set\ in\ (C(K), \mathcal{T}_Z).$$

To that end, we shall modify a construction from [72]. Let C be the collection of continuous functions

$$c : U \to \{-1, 1\} \text{ where } U = dom\ c \text{ is closed-and-open,} \tag{2}$$

Dome being the domain of c. We consider C with the discrete topology and $C^{\mathbb{N}}$is the countable product of C. Let

$$\mathcal{M} = \{ (c_1, c_2, \dots) \in C^{\mathbb{N}} : dom\ c_i \subset dom\ c_{i+1}, \\ \mu(dom\ c_i) < 1/3, c_{i+1} \mid dom\ c_i = c_i \}, \tag{3}$$

and let εbe a subspace of the product of the $space\ (C(K), \mathcal{T}_Z)$ and the space$M$, defined by

$$\varepsilon = \{(f, c_1, c_2, \dots) \in C(K) \times M : f : K \to \{-1, 1\}, f| \\ domc_i = c_i\ for\ all\ i \}. \tag{4}$$

A key element of our reasoning is the following fact.

Claim: Let $\mathcal{G}_1, \mathcal{G}_2, \dots$be open sets in E, dense in a fixed nonempty open set in ε. Then there exists a continuous function $h: H \to \{-1, 1\}$defined on a closed-and-open set in $K$with $\mu(H) \leq 1/3$, and a point $(c_1, c_2, \dots) \in \mathcal{M}$such that for any continuous f : K → {−1, 1} extending $h, (f, c_1, c_2, \dots) \in \bigcap_n \mathcal{G}_n$ .Each finite sequence $(c_1, \dots, c_r) \in C^r$ which can be extended to a point in M, cf. (3), determines a basic neighborhood in M

$$\mathcal{N}(c_1, \dots, c_r) = \{(c_1, \dots, c_r, c_{r+1}, \dots) : (c_1, c_2, \dots) \in M\}, \tag{5}$$

and a convenient base for the topology in E is defined by

$$\mathcal{N}(f, \mathcal{Z}, c_1, \dots, c_r) = \left(\mathcal{N}(f, \mathcal{Z}) \times \mathcal{N}(c_1, \dots, c_r)\right) \cap \varepsilon, \\ where\ f|\ dom\ c_r = c_r. \tag{6}$$

To check the claim, we shall define inductively continuous functions $f_n : K \to \{-1,1\}$,Sets $Z_n \in Z$, and a sequence $c_1, \dots, c_r, c_{r+1}, \dots, c_{r_2}, \dots, c_{r_n}, \dots, c_{r_{n+1}}, \dots$ of elements of C which determines a point in M, cf. (3), such that, cf. (6),

$$\emptyset \neq \mathcal{N}(f_n, Z_n, c_1, \dots, c_{r_n}) \subset \mathcal{G}_n, \qquad Z_n \subset dom\, c_{r_n}. \tag{7}$$

We shall start from picking a point $(f_1, c_1, c_2, \dots) \in \mathcal{G}_1$ and its basic neighborhood

$$\mathcal{N}(f_1, \mathcal{Z}, c_2, \dots c_k) \subset \mathcal{G}_1,$$

cf. (6). Since $\mu(domc_k) < 1/3$ and $\mu(Z_1) = 0$, one can find a closed-and-open set $V \subset K$domc_k, cf. (2), such that $Z_1 \setminus domc_k \subset V$and $\mu(domc_k \cup V) < 1/3$. Then we can define $c_{k+1}: domc_k \cup V \to \{-1,1\}$letting $c_{k+1}|\, dom\, c_k = c_k\, , c_{k+1}|V = f_1|V$ and we set $r_1 = k + 1$. This gives (7), for $n = 1$, cf. (6).

Suppose that the neighborhood$\mathcal{N} = \mathcal{N}(f_n, Z_n, c_1, \dots, c_{r_n})$ has been already defined.

Then we pick a point $(f_{n+1}, c_1, \dots, c_{r_n}, \dots) \in \mathcal{N} \cap \mathcal{G}_{n+1}$ its neighborhood $\mathcal{N}(f_{n+1}, Z_{n+1}, c_1, \dots, c_{r_n}, \dots, c_l) \subset \mathcal{G}_{n+1}$ and then, as in the first step, we define $c_{l+1}: dom\, c_l \cup W \to \{-1,1\}$, where $W \subset K\, dom\, c_l$is closedand-open, $Z_{n+1}\setminus$ domc_l⊂W and $\mu(domc_l \cup W) < 1/3$, letting$c_{l+1}: |\, dom\, c_l = c_l, c_{l+1}|W = fn + 1|W, and\ we\ set$
$r_{n+1} = l + 1$. This provides (7) for n + 1.

Having completed the construction of the sets (7), let us consider, cf. (3) and (5),

$$U = \bigcap_i dom\, c_i\ and\ c : U \to \{-1,1\}, c|\, dom\, c_i = c_i. \tag{8}$$

Since K is extremally disconnected, the closure $\bar{U}$is open and c extends continuously over $\bar{U}$. Moreover,by (3) and (8), $\mu(U \le 1/3$ and since $\mu(\bar{U} \setminus U) = 0$, we have$\mu(\bar{U}) \le 1/3$. In effect, we get a continuous
Function

$$h: \bar{U} \to \{-1,1\}, h|U = c, \mu(\bar{U}) \le 1/3. \tag{9}$$

We shall show that the function h and the point $(c_1, c_2, \dots) \in M$defined in (7) satisfy the conditions of the claim. Let $f : K \to \{-1,1\}$be an arbitrary continuous extension of h. We have to make sure that $(f, c_1, c_2, \dots) \in \bigcap_n \mathcal{G}_n$. Since $h$coincides with c_i on its domain, cf. (8), (9), so does f, and therefore $(f, c_1, c_2, \dots) \in \varepsilon$, cf. (4). Moreover, for each n, f_n coincides with c_{r_n} on its domain, cf. (6), (7), and since$Z_n \subset dom\, c_{r_n}$, cf. (7), we have$(f, c_1, c_2, \dots) \in \mathcal{N}(f_n, Z_n, c_1, \dots, c_{r_n}) \subset \mathcal{G}_n$, for each n, cf. (6).

With the claim established, the theorem follows now readily. Let us consider the projection, continuous with respect to the topology $T_Z$in $C(K)$,

$$\pi: \varepsilon \to C(K), \quad \pi\big((f, c_1, c_2, \dots)\big) = f, \tag{10}$$

and let

$$E = \left\{ f \in C(K): f : K \to \{-1,1\}, \int f\, d\mu > 0 \right\}. \tag{11}$$

Aiming at a contradiction, assume that $E$is a C-set in $(C(K), T_Z)$. Then $\pi^{-1}(E)$ is a C-set in ε, hence open modulo meager sets in ε. Therefore, there are open sets $\mathcal{G}_1, \mathcal{G}_2, \dots$,dense in some nonempty open set in ε, such that $\bigcap_n \mathcal{G}_n$is either contained in $\pi^{-1}(E)$ or it is disjoint from $\pi^{-1}(E)$. Let $h: H \to \{-1,1\}$and $(c_1, c_2, \dots) \in M$be as in the claim, and let us extend $h$to continuous functions$f_{-1}, f_1 : K \to \{-1,1\}$, setting $f_d|K \setminus H \equiv d\ for\ d \in \{-1,1\}$. Then, for any $d \in \{-1,1\}, (f_d, c_1, c_2, \dots) \in \bigcap_n \mathcal{G}_n$ and there fore either both functions f_{-1}, f_1 are in E

or both are in $C(K)\backslash E$, cf. (10). However, since $\mu(H) \leq 1/3, f_{-1}\, d\mu < 0$ *and* $\int f_1\ d\mu > 0$, and we reached a contradiction with (11).

To complete the proof it is enough to notice that the set *E*in (11) is an intersection of the set $\{f \in C(K) : \int f\, d\mu > 0\}$ and the set $\{f \in C(K) : f : K \to \{-1, 1\}\}$, closed in $(C(K), T_Z)$, hence the firstof these sets is not a C-set in $(C(K), T_Z)$, and it is not a C-set in the topology on *C(K)* generated by the Radon measures singular with respect to μ, which is weaker than T_Z.

Since, moreover, *B*us of weight $2^{\aleph_0}$, this shows that $Borel\,(C(T), norm) \neq Borel(C(T\,)$, weak), cf. [72].

It is an open problem if the Borel structures in *C(K)* coincide for separable Rosenthal compacta. Thisis true if, in addition, $K \subset B_1(\omega^\omega)$ consists of functions with at most countably many discontinuity points—in fact, as showd by Haydon, Moltó, and Orihuela [180], in this case*C (K)* admits a pointwise-Kadecnorm. It is not known whether this is still true if one retains the restriction on discontinuity points of elements of $K \subset B_1(\omega^\omega)$, but drops the separability assumption; cf. [186].

Section (5.2): Spaces of Continuous Functions

We denote by $\omega$the set of all natural numbers$\{0, 1, 2, \ldots\}$. Any $n \in \omega$ is often regarded as the set $\{0, 1, \ldots, n - 1\}$.

Let K be a compact space (all our topological spaces are Hausdorff), let C(K) be the Banish space of all continuous real-valued functions on K and let $M(K) = C(K)^*$ be space of all Radon (signed) measures on K. The M(K) is equipped with the $weak^*$topology (denoted by $w^*$for short) unless otherwise stated.

We denote by $M^+(K)$ (resp. P (K)) the subset of M (K) made up of all Radon non-negative (resp. probability) measures on K. For every $t \in K$ we denote by $\delta_t \in P(K)$ the Dirac measure at *t*. We shall write $co\Delta_K$for the convex hull of the set $\Delta_K := \{\delta_t : t \in K\}$in $M(K)$.given a set $A \subseteq M(K)$. We denote by Esq. (A) the sequential closure of A in $M(K)$, that is, the smallest subset of *M (K)* that contains A and is closed under limits of w^* −convergent sequences. The sequential closure is obtained by a transfinite procedure as follows. Define $Seq^0(A) := A$,and let $Seq^{\alpha+1}(A)$ be the set of all limits of w^*-convergent sequences in $Seq^\alpha(A)$, and let$Seq^\alpha(A) := \bigcup_{\beta<\alpha} Seq^\beta(A)$ whenever $\alpha$is a limit ordinal. Then $Seq(A) = Seq^{\omega_1}(A)$,where $\omega_1$stands for the first uncountable ordinal. The set $co\Delta_K$is w^*-dense in $P(K)$ (just apply the Hahn-Banach theorem). For an arbitrary $\mu \in P(K)$, a classical result (see [207]) states that $\mu \in Seq^1(co\Delta_K)$ if andonly if μ admits a uniformly distributed sequence, i.e. a sequence $\{t_n\}_{n\in\omega}$ in K such that $\left\{\frac{1}{n}\sum_{i<n}\delta_{t_i}\right\}_{n\in\omega}$ is ω^*-convergent to μ. There is a number of well-studied classes of compact spaces K on which *every* Radon probability measure admits a uniformly distributed sequence or, equivalently, the equality

$$Seq^1(co\Delta_K) = P(K) \tag{12}$$

Holds true. Indeed, K has such a property whenever it is metrizable, Eberlein, Rosenthal, Radon-Nikodým or a totally ordered compact line (see [206]). The space K = 2^cenjoys that property as well [201], where c stands for the cardinality of the continuum. The Stone space K of a minimally generated Boolean algebra satisfies Seq ($co\Delta_K$) = P (K) (see [196]) and, in fact, this result can be strengthen to saying that equality (12) holds.

Under the Continuum Hypothesis, we present a construction of a compact 0-dimensional space K such that

$$Seq^1(co\Delta_K) \neq Seq(co\Delta_K) = P(K)$$

(see Theorem(5.2.8) Our example has some features of an L-space constructed in [203] and related constructions given in [209]. In fact, the compact space K of Theorem (5.2.8) satisfies $Seq^1(co\Delta_K) \neq Seq^2(co\Delta_K)$ and $Seq^3(co\Delta_K) = P(K)$. Along this way, it was recently shown in [197] (without additional set-theoretic assumptions) that for every ordinal $1 \leq \alpha < \omega_1$ There is a compact space $K^{(\alpha)}$such that

$$Seq^\alpha\left(co\Delta_{K^{(\alpha)}}\right) \setminus \bigcup_{\beta<\alpha} Seq^\beta\left(co\Delta_{K^{(\alpha)}}\right) \neq \emptyset$$

and $Seq^{\alpha+1}\left(co\Delta_{K^{(\alpha)}}\right) = Seq(co\Delta_{K^{(\alpha)}}) \neq P(K^{(\alpha)})$.

Our interest on these questions is somehow motivated by their connection with the study of Baire measurability in the space C (K). Namely, if $C_\rho(K)$ (resp. $C_\omega(K)$) stands for $C(K)$ equipped with the pointwise convergence (resp. weak) topology, then the corresponding Baire σ-algebras satisfy

$$\mathrm{Ba}(C_\rho(K)) \subseteq \mathrm{Ba}(C_\omega(K)).$$

It is well-known (see [174]) that $\mathrm{Ba}(C_P(K))$ is generated by Δ_K, while $\mathrm{Ba}(C_\omega(K))$ is generated by *P(K)*. Thus, the equality

$$\mathrm{Ba}\big(C_P(K)\big) = \mathrm{Ba}\big(C_\omega(K)\big) \qquad (13)$$

Holds true whenever$Seq(co\Delta_K) = P(K)$, and this is the case for many spaces as we pointed out above. The compact space of Theorem (5.2.8) makes clear that equalities (12) and (13) are not equivalent. We pay further attention to (13) and that it fails for $K = \beta\omega$ and $K = \beta\omega \setminus\omega$ (Theorem (5.2.5) and Corollary (5.2.11).

We write $\mathcal{P}(S)$to denote the power set of any set S. Given a Boolean algebra $\mathfrak{A}$ by a 'measure' on $\mathfrak{A}$we mean a bounded finitely additive measure. The Stone space of all ultra filters on $\mathfrak{A}$ is denoted by ULT ($\mathfrak{A}$). Recall that the Stone isomorphism between $\mathfrak{A}$and the algebra Clop (ULT ($\mathfrak{A}$)) of clopen subsets of ULT ($\mathfrak{A}$) is given by

$$\mathfrak{A} \to Clop(ULT(\mathfrak{A})), \quad A \mapsto \hat{A} = \{F \in ULT(\mathfrak{A}) : A \in F\}.$$

Every measure μ on $\mathfrak{A}$ induces a measure $\hat{A} \mapsto \mu(A)$ on Clop (ULT ($\mathfrak{A}$)) which can beuniquely extended to a Radon measure on ULT ($\mathfrak{A}$) (see e.g. [211]); such Radon measure is still denoted by the same letter μ. We shall need the following useful fact about extensions of measures.

Lemma (5.2.1)[194]: ([204], [208]).*Let* $e' \supseteq \mathfrak{B}$*Be Boolean algebras and let* μ *be a* measure on$\mathfrak{B}$. Then μ can be extended to a measure$v$ *on* e'such that$inf\{v(C \vartriangle B) : B \in \mathfrak{B}\} = 0$ *for every* $C \in e'$.

A compact space K such that $Seq^1(co\Delta_K) \neq Seq(co\Delta_K) = P(K)$For the sake of the construction we first note the following two lemmas. We denote by span$\Delta_K$the linear span of $\Delta_K$in *M (K).*

Lemma(5.2.2)[194]:.*Let K* be a compact space and let$\mu \in Seq^\alpha(span\Delta_K)$*for* some$\alpha < \omega_1$. *If* $\varphi \in C(K)$and$v \in M(K)$is defined by

$$v(\Omega) := \int_{\Omega} \varphi d\mu \quad for\ every\ Borel\ set\ \Omega \subseteq K,$$

Then $v \in Seq^{\alpha}(span\,\Delta_K)$ as well. The same statement holds if span Δ_K is replaced by $M^{+}(K) \cap span\,\Delta_K$ and $\varphi \geq 0$.

Proof: We proceed by transfinite induction. The case $\alpha = 0$ being obvious, suppose that $1 \leq \alpha < \omega_1$and that the statement is valid for all ordinals $\beta < \alpha$. There is nothing to show if α is a limit ordinal, so assume that $\alpha = \xi + 1$ for some $\xi < \omega_1$. Fix a sequence $\{\mu_n\}_{n\in\omega} in\ Seq^{\xi}(span\,\Delta_K)$ which is w^*-convergent to μ. For every $n \in \omega$we define $v_n \in M(K)\ by\ v_n(\Omega) := \int_{\Omega} \varphi d\mu_n$ for every Borel set$\Omega \subseteq K$, so that $v_n \in Seq^{\xi}(span\,\Delta_K)$ by the inductive hypothesis. Clearly, for every $g \in C(K)$ we have

$$\lim_n \int_K gdv_n = \lim_n \int_K g\varphi d\mu_n = \int_K g\varphi d\mu = \int_K gdv$$

that is, $\{v_n\}_{n\in\omega} is\ w^*$-convergent to$v$. Thus $v \in Seq^{\xi+1}(span\,\Delta_K)$.

Lemma (5.2.3)[194]: Let K be a compact space and let $\mu \in Seq^{\alpha}(co\Delta_K)$ for some $\alpha < \omega_1$. If $v \in M(K)$ is absolutely continuous with respect *to* μ, then $v \in Seq^{\alpha+1}(span\,\Delta_K)$.If in addition $v \in M^{+}(K)$, then $v \in Seq^{\alpha+1}(M^{+}(K) \cap span\,\Delta_K)$.

Proof*:* Let $\varphi : K \to \mathbb{R}$be the Radon-Nikodým derivative of v with respect to μ. Fix a sequence $\{\varphi_K\}_{k\in\omega} in\ C(K)$ such that $lim_k \int_K |\varphi - \varphi_k|\, d\mu = 0$. for every $k \in \omega$we define $v_k \in M(K)\ by\ v_k(\Omega) := \int_{\Omega} \varphi_k d\mu$ for every Borel set $\Omega \subseteq K$. Since each v_k belongs to $Seq^{\alpha}(span\,\Delta_K)$ (by Lemma **(**5.2.2) and $\{v_k\}_{k\in\omega} is\ w^*$ −convergent to *v*(in fact, it is norm convergent in M(K)), it follows that $v \in Seq^{\alpha+1}(span\,\Delta_K)$. For the last assertion, just observe that φ and the $\varphi_k{}'S$ can be chosen non-negative if $v \in M^{+}(K)$.

We shall deal with the space $X := \omega \times 2^{\omega}$,where $2^{\omega} = \{0,1\}^{\omega}$the Cantor set is. For any set B ⊆X and $n \in \omega$ we write $B_{|n} := \{t \in 2^{\omega} : (n,t) \in B\}$. Let λ denote the usual product probability measure on (the Boral σ-algebra of) 2^{ω}.

We will construct algebra$\mathfrak{A} \subseteq \mathcal{P}(X)$such that the Stone space $K = ULT(\mathfrak{A})$satisfies the required properties. Let $\mathfrak{A}_0$be the algebra of subsets of X generated by the products of the form A×C where $A \subseteq \omega$is either finite or cofinite and $C \in Clop(2^{\omega})$.

Clearly,$\mathfrak{A}_0$is *admissible* in the sense of the following definition:

Definition (5.2.4)[194]: We say that a set B⊆X is *admissible* if $B_{|n} \in Clop(2^{\omega})$ for all $n \in \omega$and $lim_n \lambda(B_{|n})$ exists. In such a case, we write

$$\mu(B) := \lim_n \lambda(B_{|n})$$

We say that an algebra $\mathfrak{B} \subseteq \mathcal{P}(X)$is *admissible* if every $B \in \mathfrak{B}$is admissible.

Lemma(5.2.5)[194]: *Let* $\mathfrak{B} \subseteq \mathcal{P}(X)$be a countable admissible algebra and let $\mathcal{D} \subseteq \mathfrak{B}$. Then there is *a* set A ⊆X such that:

(i) The algebra generated by$\mathfrak{B} \cup \{A\}$is admissible*;*

(ii) For every$D \in \mathcal{D}$we have$D_{|n} \subseteq A_{|n}$for all but finitely many$n \in \omega$;

(Iii)$\mu(A) \leq \sum_{D\in\mathcal{D}} \mu(D)$.

Proof: Let $\{B_j : j \in \omega\}\ and\ \{D_j : j \in \omega\}$be enumerations of $\mathfrak{B}$and$\mathcal{D}$, respectively.

For every $k \in \omega$, we denote by $\mathfrak{B}_k \subseteq \mathcal{P}(X)$ the finite algebra generated by the collection $\{B_j : j \le k\} \cup \{D_j : j \le k\}$ and we set $\tilde{} \widetilde{D}_k := D_0 \cup \ldots \cup D_k \in \mathfrak{B}_k$. by the admissibility of $\mathfrak{B}$we can define a strictly increasing function $g : \omega \to \omega$such that for every $k \in \omega$and $n \ge g(k)$we have

$$\left|\mu(C) - \lambda(C_{|n})\right| \le \frac{1}{k+1} \qquad for\ all\ \ C \in \mathfrak{B}_k \tag{14}$$

Define a set A⊆X by declaring that

$$A_{|n} := (\widetilde{D}_k)_{|n} \text{ whenever} g(k) \le n < g(k+1)$$

and $A_{|n} := \emptyset\ if\ n < g(0)$. Clearly, A satisfies (ii).

To show (i), notice first that every element B of the algebra $\mathfrak{B}'$generated by $\mathfrak{B} \cup \{A\}$ is of the form $B = (B_j \cap A) \cup (B_i \setminus A)$ where $i, j \in \omega$. since $\mathfrak{B}$ is admissible and $A_{|n} \in Clop(2^\omega)$ for every $n \in \omega$, we have $B_{|n} \in Clop(2^\omega)$ for every $n \in \omega$ to show the admissibility of B it suffices to check that the sequences$\left\{\lambda\left(\left(B_j \cap A\right)_{|n}\right)\right\}_{n\in\omega}$ are Cauchy, because

$$\lambda(B_{|n}) = \lambda\left(\left(B_j \cap A\right)_{|n}\right) + \lambda((B_i \setminus A)_{|n}) = \lambda\left(\left(B_j \cap A\right)_{|n}\right) + \lambda((B_i)_{|n}) - \lambda\left(\left(B_j \cap A\right)_{|n}\right)$$

for every $n \in \omega$. $Fix\ \varepsilon > 0$. Since μ is a probability measure on $\mathfrak{B}$, the sequence$\left\{\mu(B_j \cap \widetilde{D}_k))\right\}_{k\in\omega}$is increasing and bounded and there is
$k_0 \in \omega$ Such that

$$\mu\left(B_j \cap \widetilde{D}_k \setminus \widetilde{D}_{k_0}\right) \le \varepsilon \text{ whenever } k \ge k_0. \tag{15}$$

Of course, we can assume further that $k_0 \ge j$ and $\frac{1}{k_0+1} \le \varepsilon$. Take any $n \ge g(k_0)$.
Then $g(k) \le n < g(k+1)$ for some k ≥ k_0, hence $(B_j \cap A)_{|n} = (B_j \cap \widetilde{D}_k)_{|n}$ and so

$$\begin{aligned}\left|\lambda\left(\left(B_j \cap A\right)_{|n}\right) - \mu\left(B_j \cap \widetilde{D}_{k_0}\right)\right| &= \left|\lambda\left(\left(B_j \cap \widetilde{D}_k\right)_{|n}\right) - \mu\left(B_j \cap \widetilde{D}_{k_0}\right)\right| \\ &\le \left|\lambda\left(\left(B_j \cap \widetilde{D}_k\right)_{|n}\right) - \left(B_j \cap \widetilde{D}_k\right)\right| + \left(B_j \cap \widetilde{D}_k \setminus \widetilde{D}_{k_0}\right) \le \frac{1}{k+1} + \varepsilon \\ &\le 2\varepsilon,\end{aligned} \tag{16}$$

By (14) and (15). It follows that $\left|\lambda\left(\left(B_j \cap A\right)_{|n}\right) - \lambda\left(\left(B_j \cap A\right)_{|m}\right)\right| \le 4\varepsilon$
Whenever $n, m \ge g(k_0)$.this s that the sequence $\left\{\lambda\left(\left(B_j \cap A\right)_{|m}\right)\right\}_{n\in\omega}$
is Cauchy.

Finally,(iii) follows from the argument above by choosing
$j \in \omega\ with\ B_j = X$.Indeed, by taking limits in (16) when $n \to \infty$we get $\left|\mu(A) - \mu\left(\widetilde{D}_{k_0}\right)\right| \le 2\varepsilon$ and so

$$\mu(A) \le 2\varepsilon + \mu\left(\widetilde{D}_{k_0}\right) \le 2\varepsilon + \sum_{i \le k_0} \mu(D_i) \le 2\varepsilon + \sum_{i\in\omega} \mu(D_i)$$

$As \varepsilon > 0$ is arbitrary, we have $\mu(A) \le \sum_{i\in\omega} \mu(D_i)$ and the proof is over.
Given an admissible algebra $\mathfrak{B} \subseteq P(X)$, we write $\mathcal{N}(\mathfrak{B})$ to denote the collection of all decreasing sequences $\{B_k\}_{k\in\omega}$ in $\mathfrak{B}$ such that $lim_k \mu\ (B_k) = 0$.

Lemma (5.2.6)[194]: Let $\mathfrak{B} \subseteq \mathcal{P}(X)$ be a countable admissible algebra, $S \subseteq \mathcal{N}(\mathfrak{B})$ a countable collection and $\varepsilon > 0$. Then there is a set A ⊆X such that:

(i) The algebra generated by $\mathfrak{B} \cup \{A\}$is admissible;

(ii) For every $\{B_k\}_{k\in\omega} \in S$there $is\ k_0 \in \omega$such that

$$(B_{k_0})_{|n} \subseteq A_{|n}\ for\ all\ but\ finitely\ many\ n \in \omega;$$

(iii) $\mu(A) \leq \omega$.

In this case we say that S is ε-captured by A.

Proof: Enumerate $S = \{\{B_k^j\}_{k\in\omega} : j \in \omega\}$. for every $j \in \omega$ we can pick $k(j) \in \omega$ smuch that $\mu(B_{k(j)}^j) \leq \varepsilon/2^{j+1}$. Now it suffices to apply Lemma **(**5.2.5) to the collection $\mathcal{D} := \{B_{k(j)}^j : j \in \omega\}$.

Lemma (5.2.7)[194]: Let$\mathfrak{B} \subseteq \mathcal{P}(X)$ be an admissible algebra containing $\mathfrak{A}_0$. Let$\{v_k\}_{k\in\omega}$be a sequence of probability measures on P(X) such that*:*

(i) Eachv_k is supported by a finite subset of X*;*

(ii) $lim_k v_k(B) = \mu(B)\ for\ every\ B \in \mathfrak{B}$.

Let $B_0 \in \mathfrak{B}$besuch that $\mu(B_0) > 0$. Then there is $A \subseteq B_0$such that the algebra generated by $\mathfrak{B} \cup \{A\}$is admissible and $\{v_k(A)\}_{k\in\omega}$does not converge to μ (A).

Proof: For every $k \in \omega$ we fix a finite set $S_k \subseteq X$ such that $v_k(S_1) = 1$. We begin by choosing two strictly increasing sequences inω, say $\{n_j\}_{j\in\omega}$and $\{k_j\}_{j\in\omega}$,such that $n_0 = k_0 = 0$and for every $j \in \omega$ we have:

(a) $v_{k+1}(R_j \cap B_0) > \mu(B_0) / 2$, where $R_j := (\omega \setminus n_j) \times 2^\omega \in \mathfrak{A}_0$;

(b) $S_{k_j+1} \subseteq n_{j+1} \times 2^\omega$.

This can be done by induction. Indeed, given $n_j, k_j \in \omega$, the conditions

$$\lim_k v_k(R_j) = \mu(R_j) = 1\ \ and\ \ \lim_k v_k(B_0) = \mu(B_0) > 0$$

Ensure the existence of$k_{j+1} > k_j$ for which (a) holds; then we choose $n_{j+1} > n_j$satisfying (b) (bear in mind that $S_{k_{j+1}}$is finite).

Fix $n \in \omega. Take\ j \in \omega$ such that $n_j \leq n < n_{j+1.}$since λ is atomless and $(B_0 \cap S_{k_{j+1}})_{|n}$ is finite, there is $C_n \in Clop(2^\omega)$ such that

$$(B_0 \cap S_{k_{j+1}})_{|n} \subseteq C_n \subseteq (B_0)_{|n} \tag{17}$$

and

$$\lambda(C_n) \leq \frac{1}{j+1} \tag{18}$$

Now, define a set $A \subseteq B_0$ by declaring that $A_{|n} := C_n$ for every $n \in \omega$.We claim that A satisfies the required properties. Note that the algebra $\mathfrak{B}'$generated by $\mathfrak{B} \cup \{A\}$ is made up of all sets of the form $(B_1 \cap A) \cup (B_2 \setminus A)$ where $B_1, B_2 \in \mathfrak{B}$. Thus, since $\mathfrak{B}$ is admissible and $A_{|n} \in Clop(2^\omega)$ for every $n \in \omega$, we also have $B_{|n} \in Clop(2^\omega)$for every $B \in \mathfrak{B}'$ and $n \in \omega$. on the other hand, (18) implies that $\lim_n \lambda(A_{|n}) = 0$,

Hence $\mu(A) = 0$ and for any $B_1, B_2 \in \mathfrak{B}$ there exists the limit

$$\lim_n \lambda\left(\left((B_1 \cap A) \cup (B_2 \setminus A)\right)_{|n}\right) = \mu(B_2)$$

This shows that $\mathfrak{B}'$is admissible.

On the other hand, we claim that for every $j \in \omega$ we have

$$R_j \cap B_0 \cap S_{k_{j+1}} \subseteq A \cap S_{k_{j+1}}. \qquad (19)$$

Indeed, take $n \in \omega$ If either $n < n_j$ or $n \geq n_{j+1}$, then $(R_j \cap B_0 \cap S_{k_{j+1}})_{|n} = \emptyset$ (bear in mind (b)). If $n_j \leq n < n_{j+1}$, then (17) implies that $(R_j \cap B_0 \cap S_{k_{j+1}})|n \subseteq (A \cap S_{k_{j+1}})_{|n}$.
This shows the inclusion (19).

It follows that for every $j \in \omega$ we have

$$v_{k_{j+1}}(A) = v_{k_{j+1}}\left(A \cap S_{k_{j+1}}\right) \geq^{(2.6)} v_{k_{j+1}}\left(R_j \cap B_0 \cap S_{k_{j+1}}\right) =$$
$$v_{k_{j+1}}\left(R_j \cap B_0\right) >^{(a)} \frac{\mu(B_0)}{2} > 0.$$

Hence the sequence $\{v_k(A)\}_{k\in\omega}$does not converge to $\mu(A) = 0$.
After those preparations we are ready for the main result.

Theorem (5.2.8)[194]: Assuming CH there is a compact space K such that

$$Seq^1(co\Delta_k) \neq Seq(co\Delta_k) = P(K).$$

Proof: Let $\{\{v_k^\xi\}_{k\in\omega} : \xi < \omega_1\}$ be the collection of all sequences of non-negative measures on $\mathcal{P}(X)$ which are supported by a finite subset of X. Weshall construct by induction an increasing transfinite collection of countable admissible algebras $\{\mathfrak{A}_\xi : \xi < \omega_1\}$ of subsets of X. We start from the algebra $\mathfrak{A}_0$ already definedand for any limit ordinal $\xi < \omega_1$ we simply set $\mathfrak{A}_\xi := \bigcup_{\eta<\xi} \mathfrak{A}_\eta$.

For the successor step of the induction, let $\xi < \omega_1$ and suppose we have already constructed the algebras $\{\mathfrak{A}_\eta : \eta \leq \xi\}$. For every $\eta \leq \xi$ we enumerate $\mathcal{N}(\mathfrak{A}_\eta)$ as $\{S(\eta,\alpha) : \eta < \omega_1\}$. Lemma (5.2.6) applied to the countable collection

$$S(\xi) := \{S(\eta,\alpha) : \eta,\alpha < \xi\} \subseteq \mathcal{N}(\mathfrak{A}_\xi)$$

ensures the existence of a set $A(\xi\ 2) \subseteq X$ such that $S(\xi)$ is (1/2)-captured by $A(\xi,2)$. Since the algebra generated by $\mathfrak{A}_\xi \cup \{A(\xi,2)\}$ is admissible, we can apply again Lemma (5.2.6) to that algebra to find a set $A(\xi,3) \subseteq X$such that $S(\xi)$ is (1/3)-captured by $A(\xi,3)$ and the algebra generated by $\mathfrak{A}_\xi \cup \{A(\xi,2), A(\xi,3)\}$is admissible. Continuing in this manner we obtain a sequence $\{A(\xi,j) : j \geq 2\}$of subsets of X such that:

(a) $S(\xi)$ is (1/j)-captured by $A(\xi,j)$ for all $j \geq 2$;

(b) The algebra $\overline{\mathfrak{A}}_\xi$generated by $\mathfrak{A}_\xi$and thefamily$\{A(\xi,j) : j \geq 2\}$ is admissible.

We now define a set $A(\xi,1) \subseteq X$by distinguishing two cases:

(A) If $\lim_k v_k^\xi(B) = \mu(B)$ for every $B \in \overline{\mathfrak{A}}_\xi$, then we can apply Lemma (5.2.6) to find a set $D_\xi \subset X\ A(\xi,2)$ such that the algebra generated by $\overline{\mathfrak{A}}_\xi \cup \{D_\xi\}$ is admissible and $\{v_k^\xi(D_\xi)\}_{k\in\omega}$does not converge to $\mu(D_\xi)$. Set $A(\xi,1) := X\ D_\xi$.

(B) Otherwise, we set $A(\xi,1) := A(\xi,2)$.

We now conclude the successor step by letting $\mathfrak{A}_{\xi+1}$ be the countable algebra generated by $\mathfrak{A}_\xi$ and the family $\{A(\xi,j) : j \geq 1\}$. Observe that $\mathfrak{A}_{\xi+1}$is admissible. Define an admissible algebra $\mathfrak{A} \subseteq \mathcal{P}(X)$ by $\bigcup_{\xi<\omega_1} \mathfrak{A}_\xi$.

Note that$\mathfrak{A}$ has thefollowing properties:

(i) For every countable collection $S \subseteq \mathcal{N}(\mathfrak{A})$ and every $\varepsilon > 0$ there is $A \in \mathfrak{A}$such that $S$is ε-captured by A;
(ii) For every sequence $\{B_k\}_{k\in\omega} \in \mathcal{N}(\mathfrak{A})$ there exists $\xi_0 < \omega_1$such that for every $\xi_0 \leq \xi < \omega_1$ and every j ≥ 1 there is $k \in \omega$ such that $(B_k)_{|n} \subseteq A(\xi, j)_{|n}$for all but finitely many $n \in \omega$.
Indeed, these facts follow from property (a) above, bearing in mind that any countable collection $S \subseteq \mathcal{N}(\mathfrak{A})$ is contained in $S(\xi_0)$ for some $\xi_0 < \omega_1$
II. Introducing the compact space K. We now consider the compact space $K = ULT(\mathfrak{A})$. Let $K^* \subseteq K$ be the set of all ultrafilters that contain no set of the form $\{n\} \times 2^\omega$. We claim that every $\mathcal{F} \in K \setminus K^*$is of the form $F_x := \{A \in \mathfrak{A} : x \in A\}$for some$x \in X$. Indeed, if $\mathcal{F}$contains $\{n\} \times 2^\omega$.

For some $n \in \omega$, then the collection $\{A_{|n} : A \in \mathcal{F}\}$is an ultrafilter on $Clop(2^\omega)$,so the intersection$\bigcap\{A_{|n} : A \in \mathcal{F}\}$consists of a single point $t \in 2^\omega$,and there fore$\mathcal{F} = \mathcal{F}_{\mathcal{X}}\ for\ \mathcal{X} := (n, t) \in X$. Since $K \setminus K^* = \bigcup_{n\in\omega} \widehat{\{n\} \times 2^\omega}$ the set $K^*$is closed in K. For any $A, B \in \mathfrak{A}$ we have

$$\hat{B} \cap K^* \subseteq \hat{A} \cap K^* \text{ whenever } B_{|n} \subseteq A_{|n} \text{ for all but finitely many } n \in \omega. \quad (20)$$

Since $\mathfrak{A}$ is admissible, for every $n \in \omega$ we have a probability measure $\mu_n$on $\mathfrak{A}$ defined by

$$\mu_n(A) := \lambda(A_{|n})$$

and $\lim_n \mu_n(A) = \mu(A)$ for every $A \in \mathfrak{A}$. Note that μ (seen as a Radon measure on K)is concentrated on K^*, because $\mu(\{n\} \times 2^\omega) = 0$ for every $n \in \omega$. We also have

$$\mu(\{\mathcal{F}\})\ 0 \text{ for every } \mathcal{F} \in K. \quad (21)$$

Indeed, fix$\varepsilon > 0$ and take any partition C of 2^ω into finitely many clopen sets with $\lambda(C) \leq \varepsilon$ for all $C \in C$. For every $\mathcal{F} \in K$ there is some $C \in C$such that $\omega \times C \in \mathcal{F}$and so $\mu(\{F\}) \leq \mu(\omega \times C) = \lambda(C) \leq \varepsilon. As\ \varepsilon > 0$ is arbitrary, this shows (21).
III. Claim. Every closed $\mathcal{G}_\delta$ set $H \subseteq K^*$ with $\mu(H) = 0$ is amortizable Indeed, it is easy to see that we can write

$$H = \bigcap_{k\in\omega} \widehat{B_k} \cap K^* \quad (22)$$

for some $\{B_k\}_{k\in\omega} \in \mathcal{N}(\mathfrak{A}$. Now let $\xi_0 < \omega_1$ be as in property I(ii) above. We shall check that the countable family $\mathcal{A} := \{\hat{A} \cap H : A \in \mathfrak{A}_{\xi_0}\}$is a topological basis of H (which implies that H is amortizable).

To this end it is sufficient to that $\widehat{A} \cap H \in \mathcal{A}$whenever $A \in \mathfrak{A}$. we proceed by transfinite induction bearing in mind that$\mathfrak{A} = \bigcup_{\xi\in\omega_1} \mathfrak{A}_\xi$. Let $\xi < \omega_1$and suppose that $\widehat{A} \cap H \in \mathcal{A}$when ever$A \in \bigcup_{\eta<\xi} \mathfrak{A}_\eta$. If either ξ is a limit ordinal or $\xi \leq \xi_0$then there is nothing to show. If ξ is of the form $\xi = \eta + 1$ for some $\eta \geq \xi_0$, set

$$e' := \{A \in \mathfrak{A}_\xi : \hat{A} \cap H \in \mathcal{A}\}.$$

Observe that $e'$is an algebra of subsets of X containing$\mathfrak{A}_\eta$. By the choice of ξ_0(bearing in mind (20)), for every $j \geq 1$ there is $k \in \omega$ such that $\widehat{B_k} \cap K^* \subseteq \setminus\widehat{A(\eta, j)} \cap K^*,\ so\ H \subseteq \setminus A(\eta, j)$ (by (22)), hence $\widehat{A(\eta, j)} \cap H = H \in \mathcal{A}$and therefore $A(\eta, j) \in \mathfrak{C}$. It follows that $\mathfrak{A}_\xi = e'$.this shows that $\widehat{A} \cap H \in \mathcal{A}$ when ever$A \in \mathfrak{A}$, as required.

IV. Claim: For every $j \in \omega$, let $H_j \subseteq K^*$ be a closed $\mathcal{G}_\delta$ set with $\mu(H_j) = 0$. Then $F := \overline{\bigcup_{J\in\omega} H_J}$ is metrizable and $\mu(F) = 0$.

Indeed, as in the previous step, for every $j \in \omega$ we can choose $\{B_k^j\}_{k\in\omega} \in \mathcal{N}(\mathfrak{A})$such that

$$H_j = \bigcap_{k\in\omega} \widehat{B_k^J} \cap K^*.$$

Fix $i \in \omega$. By property I(i), there is $A_i \in \mathfrak{A}$such that the collection $\{\{B_k^j\}_{k\in\omega} : j \in \omega\}$ is $\frac{1}{i+1}$-captured by A_i. In view of (20), for every $j \in \omega$we have

$$H_j = \bigcap_{k\in\omega} \widehat{B_k^J} \cap K^* \subseteq \widehat{A_\iota} \cap K^*$$

and so $F \subseteq \widehat{A_\iota} \cap K^*$. Since $\mu(\widehat{A_\iota}) \leq \frac{1}{i+1}$ for every$i \in \omega$, we have $\mu(F) = 0$. Moreover, since $F \subseteq H := \bigcap_{i\in\omega} \widehat{A_\iota} \cap K^*$and H is metrizable (by Claim III), it follows that F is amortizable as well.

V. Claim: For every closed separable set$D \subseteq K^*$*we have* μ(D) = 0.Indeed, let $\{\mathcal{F}_j : j \in \omega\}$be a dense sequence in D. For every j ∈ω we have$\mu(\{\mathcal{F}_j\}) = 0$ (by (21)) and so there is a closed $\mathcal{G}_\delta$ set $H_j \subseteq K^*$containing $\mathcal{F}_j$ with $\mu(H_j) = 0$. Since $\subseteq \overline{\bigcup_{J\in\omega} H_J}$, an appeal to Claim IV ensures that μ (D) = 0.

VI. Claim: The measure μ does not belong to $Seq^1(co\Delta_k)$.

Our proof is by contradiction. Suppose there is a sequence $\{\theta_k\}_{k\in\omega}$in $co\Delta_k$which is w^*-convergent to μ. For every $k \in \omega$, consider the finite set

$$I_k := \{\mathcal{F} \in K^* : \theta_k(\{\mathcal{F}\}) > 0\}$$

And let θ'_k be the Radon measure on K defined by

$$\theta'_k(\Omega) := \theta_k(\Omega \setminus K^*) for\ every\ Borel\ set\ \ \Omega \subseteq \mathrm{K}.$$

We claim that $\{\theta'_k\}_{k\in\omega}$ is w^* −convergent to μ. Indeed, the set $D := \overline{\bigcup_{k\in\omega} I_k}$satisfies $\mu(D) = 0$ (by Claim V) and so we can find such that $D \subseteq \bigcap_{i\in\omega} \widehat{B_\iota}$.Now, fix $A \in \mathfrak{A}$.We have

$$\left|\theta'_k(\hat{A}) - \theta_k(\hat{A})\right| = \theta_k(\hat{A} \cap K^*) =\leq \theta_k(\widehat{B_\iota})\text{for every } \ i,k \in \omega.$$

Bearing in mind that

$$\lim_k \theta_k(\hat{A}) = \mu(\hat{A}), \qquad \lim_k \theta_k(\widehat{B_\iota}) = \mu(\widehat{B_\iota}) \qquad and\ \lim_i \mu(\widehat{B_\iota}) = 0,$$

we get $lim_k \theta'_k(\hat{A}) = \mu(\hat{A})$. *As* $A \in \mathfrak{A}$ is arbitrary, $\{\theta'_k\}_{k\in\omega}$ is w^* −Conver Gent to μ.

On the other hand, each θ'_k is a linear combination (with non-negative coefficients) of finitely many elements of $\Delta_{k\setminus k^*} = \{\delta_{\mathcal{F}_\mathcal{X}} : x \in X\}$, *where* $\mathcal{F}_\mathcal{X} = \{A \in \mathfrak{A} : \mathcal{X} \in A\}$ (see II). Hence θ'_k comes from a non-negative finitely supported measure on $\mathcal{P}(X)$and so there is $\xi < \omega_1$such that$\theta'_k(\hat{A}) = v_k^\xi(A)$ for every $A \in \mathfrak{A}$ and $k \in \omega$. Bythe construction (see I) there is some $A \in \mathfrak{A}$ such that $\{v_k^\xi(A)\}_{k\in\omega}$does not converge to μ (A), thus contradicting the fact that $\{\theta'_k\}_{k\in\omega}$ is w∗-convergent to μ.

VII. Claim: The measure μ belongs to$Seq^2(co\Delta_k)$.

Indeed, the sequence $\{\mu_n\}_{n\in\omega}$ $is\ w^* -$convergent to μ as we pointed out in II. On theother hand, every μ_n is concentrated on the closed metrizableset $\widehat{\{n\} \times 2^\omega}$ (it is notdifficult to check that it is homeomorphic to 2^ω), hence μ_n has a uniformly distributed sequence and so $\mu_n \in Seq^1(co\Delta_k)$.

VIII. Claim: The equality $Seq^3(co\Delta_k) = P(K)$ holds.

Let $v \in P(K)$. We can write $v = v_1 + v_2 + v_3$ where $v_i \in M^+(K)$ satisfy:

(a) $v_1(\Omega) := v(\Omega \setminus K^*)$for every Borel set $\Omega \subseteq K$;

(b) $v_2$is absolutely continuous with respect to μ;

(c) $v_3$is concentrated on a Borel set $B \subseteq K^*$with $\mu(B) = 0$.

For everyn $\in \omega$we define $\theta_n \in M^+(K)\ by$

$$\theta_n(\Omega) := v\big(\Omega \cap \widehat{n \times 2^\omega}\big) \text{ for every Boral set } \Omega \subseteq K$$

Then $\theta_n \in Seq^1(M^+(K) \cap\ span\Delta_k)$,because θ_n is concentrated on the closed mortizable set $\widehat{n \times 2^\omega} = \bigcup_{k<n} \widehat{\{k\} \times 2^\omega}$. Since the sequence $\{\theta_n\}_{n\in\omega}$ $is\ w* -$ conver gent to v_1 ,we conclude that

$v_1 \in Seq^2(M^+(K) \cap\ span\ \Delta_k)$.

On the other hand, since v_2 is absolutely continuous with respect to $\mu \in Seq^2(co\Delta_k)$(see ClaimVII), we have $v_2 \in Seq^3(M^+(K) \cap\ span\Delta_k)$ by Lemma **(**5.2.3) .

Concerning v_3, note that (by the regularity of v_3) we can assume that B is of the form $B = \bigcup_{j<\omega} F_j$ for some closed sets $F_j \subseteq K^*$.Now, for every $j \in \omega$ we can find (using the regularity of μ) a closed $\mathcal{G}_\delta\ set\ H_j \subseteq K^*$such that $F_j\ \subseteq H_j\ \ and\ \ \mu(H_j) = 0$.

From Claim IV it follows that $\bar{B}$ is metrizable and so $v_3 \in Seq^1(M^+(K)\ \cap\ span\ \Delta_k)$.

Therefore, $v = v_1 + v_2 + v_3 \in Seq^3(M^+(K)\ \cap\ span\ \Delta_k)$.since$v$is a probability measure, it is not difficult to show that $v \in Seq^3(co\ \Delta_k)$ as well. This completes the proof of Theorem **(**5.2.8)

The cases of $\beta_\omega\ and\ \beta_\omega \backslash \beta$

Let K be a compact space. It is known (cf. [210]) that for every $Ba(C_p(K))$-measurable $\mu \in P(K)$ there is a closed separable set $F \subseteq K$ such that $\mu(F) = 1$. Thus, if the equality

$$Ba(C_p(K)) = Ba(C_w(K))$$

Holds true then every element of *P (K)* is concentrated on some closed separable subset of *K*. We make clear that the converse statement fails in general, since $Ba(C_p(\beta_\omega)) \neq Ba(C_w(\beta_\omega))$ (Theorem **(**5.2.5) this will be a consequence of the construction given in Theorem **(**5.2.3) below.

Recall that the asymptotic density of a set $A \subseteq \omega$ is defined as

$$d(A) := \lim_n \frac{|A \cap n|}{n}$$

Whenever the limit exists. We shall write $\mathcal{D}$ for the family of those $A \subseteq \omega$for which *d (A)* is defined. The following lemma is well-known.

Lemma(5.2.9)[194]: *If* $\{A_n\}_{n\in\omega}$is an increasing sequence in $\mathcal{D}$, then there is $B \in \mathcal{D}$such that $A_n \setminus B$is finite for every $n \in \omega$ and $d(B) = lim_n\ d(A_n)$.

Theorem (5.2.10) [194]: There is $v \in P(\beta_\omega)$ such that:

(i) $v$is of countable type, i.e. $L^1(v)$ is separable;

(ii) $v(\beta_\omega \setminus \omega) = 1$;

(iii) $v(F) = 0$for every closed separable set $F \subseteq \beta_\omega \setminus \omega$.

Proof: Let $\{t_n\}_{n\in\omega}$be a uniformly distributed sequence for the usual product probability measure λ on 2^ω. For every $E \in Clop(2^\omega)$ we define $\varphi(E) := \{i \in \omega : t_i \in E\}$,so that

$$\lim_n \frac{|\varphi(E) \cap n|}{n} = \lim_n \frac{1}{n}\sum_{i<n} 1_E(t_i) = \lambda(E),$$

Hence $\varphi(E)$ belongs to $\mathcal{D}$and $d(\varphi(E)) = \lambda(E)$.It is easy to check that

$$\mathfrak{A} \coloneqq \{\varphi(E): E \in Clop\,(2^\omega)\} \subseteq \mathcal{D}$$

is a (countable) algebra. Let $IS(\mathfrak{A})$ be the family of all increasing sequences in $\mathfrak{A}$. By Lemma (5.2.9) for every $S = \{S_n\}_{n\in\omega} \in IS(\mathfrak{A})$ we can find $B_S \in \mathcal{D}$such that $S_n \setminus B_S$ is finite for every $n \in \omega$ and $d(B_S) = lim_n\, d(S_n)$.

Let $\mathfrak{B} \subseteq \mathcal{P}(\omega)$ be the algebra generated by $\mathfrak{A} \cup \{B_S : S \in IS(\mathfrak{A})\}$. Fix any free ultra filter $\mathcal{U}$ on ω and define a probability measure μ on $\mathfrak{B}$by

$$\mu(B) \coloneqq \lim_{n\to\mathcal{U}} \frac{|B \cap n|}{n},$$

so that $\mu(B) = d(B)$ whenever $B \in \mathfrak{B} \cap \mathcal{D}$. Observe that the family

$$\mathfrak{B}_0 \coloneqq \{B \in \mathfrak{B}: \inf\{\mu(B \,\Delta\, A): A \in \mathfrak{A}\} = 0\}$$

is an algebra containing $\mathfrak{A}$. We claim that $B_S \in \mathfrak{B}_0$ for every $S = \{S_n\}_{n\in\omega} \in IS(\mathfrak{A})$.

Indeed, fix $n \in \omega$and observe that, since $S_n \setminus B_S$is finite and $S_n \in \mathcal{D}$,, we have $S_n \cap B_S \in \mathcal{D}$and $d(S_n \cap B_S) = d(S_n)$. Now, since $B_S \in \mathcal{D}$we also have $B_S \setminus S_n \in \mathcal{D}$and

$$d(B_S \setminus S_n) = d(B_S) - d(S_n \cap B_S) = d(B_S) - d(S_n),$$

Hence $\mu(B_S \vartriangle S_n) = \mu(S_n \setminus B_S) + \mu(B_S \setminus S_n) = d(B_S) - d(S_n)$.

Bearing in mind that $d(B_S) = lim_n d(S_n)$, we conclude that$B_S \in \mathfrak{B}_0$, as required.

It follows tha t$\mathfrak{B}_0 = \mathfrak{B}$. Using Lemma (5.2.1), we extend μ to a probability measure $v\ on\ \mathcal{P}(\omega)$ so that$inf\{v(C \vartriangle A) : A \in \mathfrak{A}\} = 0$ for every $C \subseteq \omega$. Observe that v(seen as a Radon measure on β_ω) has countable type (because A is countable).

In order to check that $v$is concentrated on $\beta_\omega\setminus\omega,\ fix\ n \in \omega$and take any$\varepsilon > 0$. Choose a partition $2^\omega = \bigcup_{i=1}^{P} E_i$such that each $E_i \in Clop(2^\omega) and\ \lambda(E_i) \le \varepsilon$.Then $\omega = \bigcup_{i=1}^{P} \varphi(E_i)$, each $\varphi(E_i) \in \mathfrak{A}$and $v(\varphi(E_i)) = \mu(\varphi(E_i)) = d(\varphi(E_i)) = \lambda(E_i) \le \varepsilon$.Since $n \in \varphi(E_i)$ for some i, we have $v(\{n\}) \le v\big(\varphi(E_i)\big) \le \varepsilon.\ As\ \varepsilon > 0$ is arbitrary, we get $v(\{n\}) = 0$. It follows that $v(\beta_\omega\setminus\omega) = 1$.

Finally, take any closed separable set $F \subseteq \beta_\omega\setminus\omega$and let $\{\mathcal{F}_n\}_{n\in\omega}$be a dense sequence in F. Fix $\varepsilon > 0$. As in the previous paragraph, for every $k \in \omega$we can find a partition of ωinto finitely many elements of $\mathfrak{A}$having asymptotic density less than $\varepsilon/2^{k+1}$;one of those elements, say T_k, belongs to $\mathcal{F}_k$,. Set

$$S_n \coloneqq \bigcup_{k\le n} T_k \text{for every } n \in \omega$$

so that $S = \{S_n\}_{n\in\omega} \in IS(\mathfrak{A})$.We have $\mathcal{F}_n \in \widehat{B_S}$ for every $n \in \omega$, because $S_n \setminus B_S$is finite and $S_n \in \mathcal{F}_n \in \beta_\omega\setminus\omega$. Hence $F \in \widehat{B_S}$. Since

$$d(S_n) \le \textstyle\sum_{k\le n} d(T_k) < \sum_{k<n} \frac{\varepsilon}{2^{k+1}} < \varepsilon \text{for every } n \in \omega,$$

it follows that $v(F) \le v\big(\widehat{B_S}\big) = \mu(B_S) = d(B_S) = \lim_n d(S_n) \le \varepsilon$.

As $\varepsilon > 0$is arbitrary, we get $v(F) = 0$. The proof is over.
Bearing in mind the comments at the beginning, Theorem (5.2.10) gives immediately the following:

Corollary(5.2.11)[194]: $.Ba(C_p(\beta_\omega \backslash \omega)) \neq Ba(C_\omega(\beta_\omega \backslash \omega))$.

We arrive at the main result.

Theorem (5.2.12)[194]: $Ba(C_p(\beta_\omega \backslash \omega)) \neq Ba(C_\omega(\beta_\omega))$.

Proof: Let $v \in P(\beta_\omega)$ be the measure of Theorem (5.2.10). We shall show that $v$is not $Ba(C_\omega(\beta_\omega))$-measurable by contradiction. Suppose v is $Ba(C_p\beta_\omega))$-measurable and fix a countable set $I \subseteq \beta_\omega$ such that v is measurable with respect to the σ-algebra Σ on *C(K)* generated by $\{\delta_{\mathcal{F}} : F \in I\}. Set\ F := \overline{I \backslash \omega} \subseteq \beta_\omega \backslash \omega$, so that $v(F) = 0$.
Thus, there is $A \subseteq \omega$ with $v(A) > 0$ such that $\hat{A} \cap F = \emptyset$.

We can define a measure m on $\mathcal{P}(A)\ by\ m(B) := v(B)$ for every $B \subseteq A$. We claim that m is Borel measurable as a function on $\mathcal{P}(A)$ (naturally identified with 2^A).

Indeed, just observe that the function

$$\emptyset : \mathcal{P}(A) \longrightarrow C(K), \qquad \emptyset(B) := 1_{\hat{B}},$$

is Borel-Σ-measurable, because $\delta_{\mathcal{F}} \circ \emptyset = 0$ for every $\mathcal{F} \in I \backslash \omega$ (bear in mind that $\hat{A} \cap (I \backslash \omega) = \emptyset$).Since in addition m vanishes on finite sets and $m(A) > 0$, an appeal to [202] (cf. [201]) ensures that m (seen as a Radon measure on βA) has uncountable type, which Contradicts the fact that v has countable type.

While P-points do exist under Martin's axiom and in many standard models of ZFC, consistently there are no *P*-points [212]. Moreover, consistently there are no measures on $\mathcal{P}(\omega)$ extending asymptotic density and having property (AP) [205].

Recall that *C(K)* is called a Grothen dieck space if every w^*-convergent sequence in *M(K)* is necessarily weakly convergent (see e.g. [198]). The spaces $C(\beta_\omega)$and $C(\beta_\omega \backslash \omega)$ are examples of Grothendieck spaces. Our motivation for Problem (5.2.17) comes from the results and the following fact:

Proposition (5.2.13)[194]: If K is infinite and *C(K)* is a Grothendieck space, then

$$Seq(co\Delta_k) \neq P(K).$$

Proof: We first claim that every element of $Seq(co\Delta_k)$ is concentrated on a countable subset of *K*. Indeed, let $\{\mu_n\}_{n\in\omega}$ be any w^*-convergent sequence in *P(K)*, where each μ_n is concentrated on a countable set $C_n \subseteq K$, and write $\mu \in P(K)$ to denote its limit. Since *C (K)* is Grothen dieck, the sequence $\{\mu_n\}_{n\in\omega}$ converges to μ weakly in *M (K)* and so

$$\mu\left(K \backslash \bigcup_{k\in\omega} C_k\right) = \lim_n \mu_n\left(K \backslash \bigcup_{k\in\omega} C_k\right) = 0,$$

therefore μ is concentrated on a countable set. This shows the claim.

Since *C(K)* is Grothendieck, it has no complemented copy of c_0(cf. [198]), hence *K* is not scattered (see e.g. [200]) and so there are elements of *P(K)* which are not concentrated on a countable subset of *K*, [211]. It follows that $Seq(co\Delta_k) \neq P(K)$.

Corollary (5.2.14)[260]: $Ba\left(C_p^2\left(\frac{\beta^2_{\omega^2-1}}{\omega^2-1}\right)\right) \neq Ba(C^2_{\omega^2-1}(\beta^2_{\omega^2-1}))$.

Proof: Let $v \in P(\beta^2_{\omega^2-1})$ be the measure of Theorem (5.2.10). We shall show that v is not $Ba(C^2_{\omega^2-1}(\beta^2_{\omega^2-1}))$-measurable by contradiction. Suppose v is $Ba(C^2_p\beta^2_{\omega^2-1}))$-measurable

and fix a countable set $I \subseteq \beta^2_{\omega^2-1}$ such that v is measurable with respect to the σ-algebra Σ on $C(K)$ generated by $\{\delta_{\mathcal{F}}: F^2 \in I\}$. Set $F^2 := \overline{I\ (\omega^2-1)} \subseteq \beta^2_{\omega^2-1} \backslash (\omega^2-1)$, so that $v(F^2)=0$.
Thus, there is $A^2 \subseteq \omega^2-1$ with $v(A^2)>0$ such that $\hat{A}^2 \cap\ F^2=\emptyset$.
Corollary (5.2.15)[260]: If K^2 is infinite and $C(K^2)$ is a Grothendieck space, then

$$Seq(co\Delta_{k^2})\ \neq P(K^2).$$

Proof: We first claim that every element of $Seq(co\Delta_{k^2})$ is concentrated on a countable subset of K^2. Indeed, let $\{\mu_n\}_{n\in\omega}$be any w^*-convergent sequence in $P(K^2)$, where each $\mu_n$is concentrated on a countable set $C_n \subseteq K^2$, and write $\mu \in P(K^2)$ to denote its limit. Since $C(K^2)$ is Grothen dieck, the sequence $\{\mu_n\}_{n\in\omega}$ converges to μ weakly in $M\ (K^2)$ and so

$$\mu\left(K^2 \setminus \bigcup_{k^2\in\omega} C_{k^2}\right)=\lim_n \mu_n\left(K^2 \setminus \bigcup_{k^2\in\omega} C_{k^2}\right)=0,$$

therefore μ is concentrated on a countable set.

Section (5.3): Banach Space

Given a compact space K, by C (K) we denote the Banach space of continuous real-valued functions K, equipped with the standard supremum norm. If $k=\beta\omega$,the ˇ Cech-Stone compactification of the space ω of natural numbers, then C ($\beta\omega$) is isometric to the classical Banach space l_∞.

One can consider three natural topologies on C (K): $\tau_p \subseteq weak \subseteq norm$, where is $\tau_p$the topology of pointwise convergence. Consequently, one has three corresponding Borel σ-algebras

$$\text{Borel}(C(K)\tau_\rho), \subseteq \text{Borel}(\text{C (K), weak}) \subseteq \text{Borel}(\text{C(K), norm}).$$

Those three σ-algebras are equal for many classes of nonmetrizable spaces K, this is the case for all spaces K such that the space C (K) admits the so called
Pointwise Kadec renorming, see [175] and [190], we also refer to [170] for some comments concerning coincidence of these σ-algebras.
On the other hand, Talagrand [69] showd that

$$Borel(C(\beta\omega), weak)\ \neq\ Borel(C(\beta\omega), norm)\ .$$

Marciszewski and Pol [170] showed that

$$\text{Borel}(C(s), \tau_p)\ \neq\ \text{Borel}(C(s), weak)$$

For S being the Stone space of the measure algebra. Since, for the space S, the Banach spaces $C(S)$ and $C(\beta\omega)$ isomorphic, it follows that $C(S)$ has three different Borel structures. Let us note that the Borel structures in function spaces

$$(C(S), \tau_p)\ \text{, and}\ (C(\beta\omega), \tau_p)$$

Are essentially different.
We show that

$$\text{Borel}(C(\beta\omega), \tau_p) \neq \text{Borel}(C(\beta\omega), weak)\ ;$$

our result and Talagrand's theorem mentioned above imply that, even though $\beta\omega$ is separable, the space $C(\beta\omega)$ possesses three different Borel structures as well.

Proving the main result, stated below as Theorem (5.3.10), we build on ideas from [170] and show that in fact there is a measure $\hat{\mu} \in C(\beta\omega)^*$ which is not point wise Borel measurable.

Recall that, if $\varphi : K \to L$ is a continuous surjection, then the map f$\mapsto$ $f \circ \varphi$ defines an embedding of $C(L)$ into $C(K)$ with respect to the norm, weak, and point wise topologies. Since $\beta\omega$ is a continuous image of ω^*, it follows from Theorem(5.3.10) that for $\omega^* = \beta_\omega / \omega$ one also has

$$\text{Borel}\,(C(\omega^*), \tau_p) \neq \text{Borel}\,(C(\omega^*), weak).$$

This result was obtained in [170] under some additional set-theoretic assumption.

We show that no sequence of pointwise Borel sets separates points of $C(\omega^*)$. Contains some remarks concerning σ-fields of Baire sets in function spaces on $\beta\omega$ and ω^*.

We shall consider only nonnegative, finite measures. We will use the well-known fact that any finitely additive measure μ on $(\omega, \rho(\omega))$ corresponds to auniquely determined Radon measure $\hat{\mu}$ on $\beta\omega$ such that $\mu(\mathrm{A}) = \hat{\mu}(\overline{\mathrm{A}})$, for any $\mathrm{A} \in \rho(\omega)$, where $\overline{\mathrm{A}}$ is the closure of A in $\beta\omega$, cf. [177].

We consider only measures μ on ω vanishing on singletons; then for the corresponding measures $\hat{\mu}$ on $\beta\omega$, we have $\hat{\mu}(\omega) = 0$, and we may as well treat such measures $\hat{\mu}$ as being defined on $\omega *$.

The following auxiliary result can be found in [215].

Proposition (5.3.1)[213]: If $(G_n)_n$ is a sequence of dense open subsets of 2^ω then there is a sequence $(I_n)_n$ of pair wise disjoint finite subsets of ω and a sequence of functions $\varphi_n : I_n \to 2$ such that $\chi \in \bigcap_n G_n$ for every $\chi \in 2^\omega$ for which the set $\{ n \in \omega : \chi / I_n = \varphi_n \}$ is infinite.

Proposition (5.3.2)[213]: No nonzero measure on ω, vanishing on singletons, is measurable with respect to the σ-algebra of subsets of 2^ω having the Baire property.

Proof: Suppose, towards a contradiction, that μ, treated as a function on 2^ω, is measurable with respect to the σ-algebra of subsets of 2^ω having the Baire property. Without loss of generality, we can assume that $\mu(\omega)$. The inverse image μ^{-1}(S) of any Borel subset S of the unit interval [0, 1] is a tail-set with the Baire property, hence, by 0 –1 Law (see [217]) is either meager or comeager. Observe that there exist (necessarily unique) $t \in [0,1]$ such that $\mu^{-1}(\tau)$ is comeager. Indeed, if $\mu^{-1}(1)$ is comeager, then we are done. Otherwise, we can define inductively a sequence of integers $\kappa_n \le 2^n - 1$, such that $\mu^{-1}([k_n / 2^n, (k_n + 1)/2^n))$is comeager for n $n \in \omega$. Then the required t is a unique element of $\bigcap_{n\in\omega} [\kappa_n / 2^n, (\kappa_n + 1)/2^n)$.

The map h: $\rho(\omega) \to \rho(\omega)$, defined by $h(A) = \omega \setminus A$, is a homeomorphism of $\rho(\omega)$ Such that h $(\mu^{-1}((\tau)) = \mu^{-1}(1-\tau)$. There fore $t = 1 - t$, and t $t = 1 \setminus 2$.

By Proposition (5.3.1), we have functions $\varphi_n : I_n \to 2$ defined on pairwise disjoint finite sets I_n such that $\mu(A) = 1/2$ whenever χ_A agrees with infinitely many φ_n 'S .

Let N_1, N_2, N_3 be a partition of! Consisting of infinite sets and let

$$\mathrm{B}_i = \bigcup \{\{\kappa : \varphi_n(\kappa) = 1\} : n \in N_i\},$$

$i=1,2,3$. Then, for each $i \le 3, \mu(B_i)=1/2$ and the sets B_i are pairwise disjoint,a contradiction. For any subset A of ω we write

$$\bar{d}(A)=\limsup_n \frac{|A \cap n|}{n},$$

For the outer asymptotic density of a set A and

$$d(A)=\lim_n \frac{|A \cap n|}{n},$$

Whenever the set A has the asymptotic density, i.e. when the above limit exists.

Given a bounded sequence $(\chi_n)_{n\in\omega}$ and an ultrafilter $\wp \in \omega^*$, by $\lim_\wp \chi_n$ we denote the $\wp \lim it$ of (χ_n). For any ultrafilter $\wp \in \omega^*$, we define the measure $d_\wp$ on ω by the formula

$$d_\wp(A)=\lim_\wp \frac{|A \cap n|}{n},$$

For $A \subseteq \omega$, cf. [177] or [195].

Lemma(5.3.3)[213]: For any ultra filter $\wp \in \omega^*$ and any $\varepsilon > 0$, there exists a set $A \in \wp$ having asymptotic density and such that $d(A) \prec \varepsilon$.

Proof: Take $n \ge 1$ such that $1/n \prec \varepsilon$ and put $A_k=\{ni+k : i \in \omega\}$ for $k=0,1,...,n-1$. Then there exists $k \prec n$ such that $A_K \in \wp$. Clearly, $d(A)=1/n$.

We also recall the following standard fact concerning the outer density. Here, for $A, B \subseteq \omega, A \subseteq *B$ denotes, as usual, that $A \setminus B$ is finite, and we denote by $A*$ the set $\bar{A} \setminus A$, where $\bar{A}$ is the closure of A in $\beta\omega$.

Lemma (5.3.4)[213]: Let $\varepsilon > 0$ and $A_n \subseteq \omega, n \in \omega$, be such that $A_n \subseteq A_{n+1}$ and $\bar{d}(A_n) < \varepsilon$ for every $n \in \omega$. Then there exists $A \subseteq \bigcup_{n\in\omega} A_n$ such the $\bar{d}(A \le \varepsilon)$ and $A_n \subseteq^* A$ for every $n \in \omega$.

Proof: By the definition of $\bar{d}$ we have

$$\forall n \in \omega \exists k_n \in \omega \forall k_n \frac{|A_n \cap k|}{k} < \varepsilon .$$

Without loss of generality, we can assume that the sequence (k_n) is increasing. We define.

$$A = \bigcup_{n\in\omega} A_n \cap (k_{n+1} \setminus k_n) \subseteq \bigcup_{n\in\omega} A_n .$$

Since $A_n \subseteq A_{n+1}$, we have $(A_n \setminus k_n) \subseteq A$, and therefore $A_n^* \subseteq A^*$ for every $n \in \omega$.

For any $k > k_0$, we have $k \in k_{n+1} \setminus k_n$, for some $n \in \omega$, and $A \cap k \subseteq A_n \cap k$, therefore $|A \cap k| / < \varepsilon$, and consequently $\bar{d}(A) \le \varepsilon$.

Obviously, for the sets A_n and A as in the above lemma, we have $\bigcup_{n\in\omega} A_n^* \subseteq A^*$ and $d_\wp(A) \le \varepsilon$ for any ultra filter $\wp \in \omega^*$. Let us note, however, that this does not necessarily mean that for every increasing sequence $A_0 \subseteq A_1 \subseteq ... \subseteq \omega$ such that $d_\wp(A) < \varepsilon$ there is A almost containing every A_n and such that $d_\wp(A) \le \varepsilon$.

Measures on $P(\omega)$ with such an approximation property may fail to exist, see [216] for details.

Corollary (5.3.5)[213]: For any ultrafilter $\wp \in \omega^*$, the measure b $\hat{d}_\wp$ vanishes on separable subsets of ω^*.

Proof: Let X be a subset of ω^* contained in the closure of a set $\{F_n : n \in \omega\} \subseteq \omega^*$. Fix $\varepsilon > 0$. For any $n \in \omega$, by Lemma**(**5.3.3) , we can pick $B_n \in F_n$ with $\bar{d}(B_n) < \varepsilon / 2^{n+1}$. Then, for $A_n = \bigcup_{k \leq n} B_k$, we have $\bar{d}(A_n) < \varepsilon$, and we can apply Lemma **(**5.3.4) for the sequence (A_n) , obtaining the set A satisfying $\bar{d}(A_n) < \varepsilon$. For any $n \in \omega$, we have $B_n \subseteq A_n \subseteq^* A$, hence $A \in F_n$. Therefore the closure in ω *.of the set $\{F_n : n \in \omega\}$ is contained in A^*, and $\hat{d}_{\wp}(X) \leq \hat{d}_{\wp}(A^*) \leq \varepsilon$. Since ε was arbitrarily chosen, it follows that $\hat{d}_{\wp}(X) = 0$.

Let $\wp$ be a fixed ultra filter from ω ∗ and let $\mu = d_{\wp}$ be the measure on $P(\omega)$.

We write $\hat{\mu}$ for the corresponding Radon measure on $\beta(\omega)$.

Then $\hat{\mu}$ is a continuous functional on $C(\beta\omega)$ so in particular $\hat{\mu}$ is measurable with respect to the σ -algebra of weakly Borel subsets.

We shall show that the measure $\hat{\mu}$ is not point wise Borel measurable and in this way conclude the main result. The approach presented below builds on the technique developed by Burke and Pol [72] and Marciszewski and Pol [170].

We need to fix several pieces of notation. For a set X, by $[X]^{<\omega}$ we denote the family of all finite subsets of X , and $X^{<\omega}$ stands for the set of all finite sequences of elements of X . Given sequences $s,t \in X^{<\omega} s \cap t$, sat denotes their concatenation.

For functions f and f g , $f < g$ means that the domain dom(f) of f is contained in the domain of g and $g \setminus dom(f) = f$. We also use this notation for sequences, treating them as functions.

Writing 2 = {0, 1}, we denote by $C_\rho(\beta\omega, 2)$ the space of all continuous functions $f : \beta\omega \to 2$ equipped with the pointwise topology.

In the sequel we consider some subsets of $(P(\omega))^2 = P(\omega) \times P(\omega)$; a typical element of such a set is a pair $C = (A, B)$, where $A, B \subseteq \omega$. Given some $C_i \in (P(\omega))^2$, we shall use the convention for elements that every C_i can be written as $C_i = (A_i, B_i)$.

Let $e^{\sim}$ be a subset of $(P(\omega) \times P(\omega))^{\omega}$ of those sequences $c = (c_0, c_1, \ldots)$ for which the following conditions are satisfied for every i:

(i) $A_{i \subseteq} A_{i+1}, B_i \subseteq B_{i+1}, A_i \cap B_i = \emptyset$;

(ii)$\bar{d}(A_i), \bar{d}(B_i) < 1/6$.

We moreover denote by $\mathfrak{S}$ the set of all finite sequences from $(P(\omega) \times P(\omega))^{<\omega}$ satisfying.

Given $f \in C(\beta\omega), i \in \{0,1\}$, and $A \subseteq \omega$, we write $f \setminus A \simeq i$ if the equality $f(x) = i$ holds for $\hat{\mu}$ -almost all $x \in A^*$.

We equip $P(\omega) \times P(\omega)$ with the discrete topology and é with the product topology inherited from $P(\omega) \times P(\omega)^{\omega}$. Finally, we define a topological space $\mathbb{E}$ that is crucial for our considerations as follows

$\mathbb{E}$={$(f, c) \in C_p(\beta\omega, 2) \times$ é : $f \setminus A_n \simeq 0$, $f \setminus B_n \simeq 1$ for every n };

here $c = (c_0, c_1, c_2 \ldots)$ and $C_i = (A_i, B_i)$.

Let $\mathfrak{I}$ be the set of all pairs

$$z = (z(0), z(1)) \in [\omega^*]^{<\omega} \times [\omega^*]^{<\omega},$$

such that $z(0) \cap z(1) = \emptyset$. For $z, z' \in \mathfrak{I}$ we write $z \subset z'$ to denote that $z(0) \subseteq z'(0)$ and $z(1) \subseteq z'(1)$. Basic open neighborhoods in $\mathbb{E}$ are of the form $N(\sigma, z, s)$, where $\sigma \in 2^{<\omega}$, $z \in \mathfrak{I}, s \in \mathfrak{S}$, , and $N(\sigma, z, s)$ is the set of all $(f, c) \in \mathbb{E}$ such that

(a) $f(x) = i$ For every $x \in z(i), i = 0,1$;

(b) $\sigma \prec f$ and $s \prec c$.

Note that every set of the form $N(\sigma, z, s)$ is nonempty, since $\hat{\mu}$ vanishes on singletons.

Let us say that $s \in \mathfrak{S}$ captures $z \in \mathfrak{I}$ if, writing $s = t \frown (A, B)$, we have $z(0) \subseteq A^*$ and $z(1) \subseteq B^*$.

Lemma (5.3.7)[213]: Every basic open set $N(\sigma, z, s)$ in $\mathbb{E}$ contains a neighborhood $N(\sigma, z, s')$ where s' captures z.

Proof: Indeed, if (A, B) is the final pair in s then for any $\varepsilon > 0$, using Lemma**(**5.3.3) we can find sets $C, D \subseteq \omega$ of asymptotic densit $y > \varepsilon$ and such that $z(0) \subseteq C^* z(1) \subseteq D^*$, $z(1) \subseteq D^*$. Then we can put $s' = s \frown (A', B')$, where $A' = A \cup C, B' = B \cup D$ and ε is small enough.

Lemma (5.3.8)[213]: Let $N(\sigma, z, s)$ be a basic open set in $\mathbb{E}$, where $\sigma \in 2^l . IfG$ is a dense open subset of $N(\sigma, z, s)$ then, for every $K \geq 1$, there are m > $k, z' \in \mathfrak{I}$ with $z \sqsubset z'$, $\in \mathfrak{S}$ with $s \prec s'$ s, and a function

$$\varphi : I = \{i : k \leq i < m\} \to 2,$$

such that for every $\tau \in 2^{k-1}$

$$N(\sigma \frown \tau \frown \varphi, z', s') \subseteq G.$$

Proof: Given $\tau_0 \in 2^{K-1}$, we have $N(\sigma \frown \tau_0, z, s) \cap G \neq \emptyset$ so for some interval $I_1 = \{i : k \leq i < m_1\}$ and $\varphi_1 : I_1 \to 2$ there are $z_1 \sqsupset z$ and $s_1 > s$ such that

$$N(\sigma \frown \tau, \varphi, z', s') \subseteq G.$$

Take another $\tau_1 \in 2^{K-1}$. Apply the same argument for $N(\sigma \frown \tau_1 \frown \varphi_1, z_1, s_1)$. It is clear that we arrive at the conclusion after examining all $\tau \in 2^{K-1}$.

Lemma(5.3.9)[213]: Let $(G_n)_{n \in \omega}$ be a decreasing sequence of open subsets of E such that $G_0 \neq \phi$ and every G_n is ense in G_0. Then there exist a sequence $c = ((A_n, B_n))_{n \in \omega} \in e'$, sets $A, B \subseteq \omega$, countable sets $Z(0), Z(1) \subseteq \omega^*$, and a sequence $\varphi_n : I_n \to 2$ of functions defined on pairwise disjoint finite sets $I_n \subseteq \omega$ such that

(i) $\bigcup_{n \in \omega} A_n^* \subseteq A^*, \bigcup_{n \in \omega} B_n^* \subseteq B^*, A \cap B = \phi$;

(ii) $\mu(A), \mu(B) \leq \frac{1}{6}$;

(iii) $Z(0) \subseteq A^*, Z(1) \subseteq B^*$;

(iv) $forevery\ f \in C_p(\beta\omega, 2) satisfying$

- $f \setminus A \simeq 0, f \setminus B \simeq 1$,
- $f \setminus Z(i) = i for i = 0,1$,
- $f \setminus I_0 = \varphi_0$,
- $f \setminus I_n = \varphi_n\ for \inf initely\ n \geq 1$,

we have $(f, c) \in \bigcap_{n \in \omega} G_n$.

Proof: Fix a basic neighborhood $N(\sigma_0, z_0, s_0) \subseteq G_0$; by Lemma**(**5.3.7) we can assume that s_0 captures z_0. Take k_0 such that $\sigma \in 2^{k_0}$, set $I_0 = \{0, \ldots, k_0 - 1\}$ and $\varphi_0 = \sigma_0$.

We shall define inductively natural numbers $k_0 < k_1 < k_2 < \ldots$, functions $\varphi_n : I_n = \{i : k_{n-1} \le i < k_n\} \to 2$, pairs $z_n \in \Im$ with $z_0 \subset z_1 \subset \ldots$, and sequences $s_0 \prec s_1 \prec \ldots$. in $\mathfrak{S}$ such that for every $n > 0$

$-s_n$ captures z_n;

– for every $\tau \in 2^{k_n - k_0}$ we have $N(\varphi_0 \cap \tau \cap \varphi_{n+1}, z_{n+1}, s_{n+1}) \subseteq G_{n+1}$.

Having $k_n, \varphi_n, \ldots$ defined, we make the inductive step using Lemma (5.3.8)for the neighborhood $N(\varphi_0, z_n, s_n)$ with $G = G_{n+1} \cap N(\varphi_0, z_n, s_n), l = k_o$, and $k = k_n$ and we use m, z', and s' given by this lemma to define k_{n+1}, z_{n+1}, and s_{n+1}. We complete our choice applying Lemma (5.3.7).

The sequence $s_n \in \mathfrak{S}$ defines the unique element $c = ((A_n, B_n))n \in \omega \in e'$; we take A, B applying Lemma (5.3.4) to sequences $(A_n)_n$ and $(B_n)_n$ (see also the remark following the proof of Lemma (5.3.4). We put $Z(0) = \bigcup_n z_n(0)$ and $Z(1) = \bigcup_n z_n(1)$; note that $Z(0) \subseteq A^*$ and $Z(1) \subseteq B^*$.

Now, if f satisfies (iv) then $(f, c) \in G_n$, for infinitely many n, so $(f, c) \in \bigcap_n G_n$.

Theorem(5.3.10)[213]: The measure $\hat{\mu}$ is not measurable with respect to the point wise Borel sets in $C(\beta\omega)$. In particular,

$$Borel(C(\beta\omega), \tau_p) \ne Borel(C(\beta\omega), weak) .$$

Proof: Suppose otherwise; then

$$F_0 = \{f \in C_p(\beta\omega, 2) : \int f d\hat{\mu} < 1/2\},$$

is pointwise Borel in $Cp(\beta\omega, 2)$. Let $F =_1 C(\beta\omega, 2) \setminus F_0$.

Let $\pi : \mathbb{E} \to C_P(\beta\omega, 2)$ denote the projection onto the first axis. It follows that the sets $\pi^{-1}(F_i)$ are Borel in $\mathbb{E}$, so both $\pi^{-1}(F_0)$ and $\pi^{-1}(F_1)$ have the Baire property in $\mathbb{E}$. Therefore, for some $i \in \{0,1\}$, there is a decreasing sequence $(G_n)_n$ of open sets in $\mathbb{E}$, where $G_0 \ne \phi$, every G_n is dense in G_0 and $\bigcap_n G_n \subseteq \pi^{-1}(F_i)$.Take $c \in e'A, B \subseteq \omega, Z(0), Z(1), \varphi_n : I_n \to 2$ as in Lemma (5.3.9)

Let $\Re$ be an uncountable almost disjoint family of infinite subsets of ω. For $R \in \Re$ let

$$I_R = I_0 \cup \bigcup_{n \in R} I_n ;$$

Then the family $\{I_R : R \in \Re\}$ is almost disjoint too. Therefore there is $R \in \Re$ such that

– $(Z(0) \cup Z(1)) \cap I_R^* = \phi$, and

– $\hat{\mu}(I_R) = 0$.

Set $A' = A \setminus I_R, B' = B \setminus I_R$. Take any function $f \in C(\beta\omega, 2)$ such that $f \equiv 0$ on A', $f \equiv 1$, on B' and f is defined on I_R so that $f \setminus I_n = \varphi_n$ for $n \in R \cup \{0\}$.

Then $f \equiv 0$ on $Z(0)$ and $f \equiv 1$ on $Z(1), f \setminus A \simeq 0, f \setminus B \simeq 1$. It follows from Lemma (5.3.9) that $(f, c) \in \bigcap_n G_n \subseteq \pi^{-1}(F_i)$.

On the other hand, f can be freely defined on the set

$$D = \omega \setminus (A' \cup B' \cup I_R),$$

Where $\mu(D) \ge 2/3$, so $\int f d\hat{\mu}$ can take values less than $1/2$ and greater than $1/2$, a contradiction.

Let us recall that in a topological space X, the elements of the smallest σ -algebra in X containing open sets and closed under the Sousing operation are called C-sets, cf. [82]. The

C-sets are open modulo meager sets and any preimage of a C-set under a continuous map is a C-set.

Theorem(5.3.11)[213]: No countable family of C-sets separates the functions in the space $\left(C(\omega^*),\tau_p\right)$.

Corollary(5.3.12)[213]: There is no Borel-measurable injection $\varphi=\left(C(\omega^*),\tau_p\right)\to\left(C(\beta\omega),\tau_p\right)$.

We keep here a part of the notation introduced; in particular, we will use the space é and the sets $\mathfrak{S},\mathfrak{I}$.

By $C_p(\omega^*,2)$ we denote the subspace of $\left(C(\omega^*),\tau_p\right)$ consisting of 0 -1-valued functions. The role of the space in $\mathbb{E}$ will be played by the following space

$\mathbb{F}=\left\{(f,c)\in C_p(\omega^*,2)\times e' : f \setminus A_n^*\equiv 0,\ f \setminus B_n^*\equiv 1\ for\,every\,n\right\}$;

Where $c=(c_0,c_1,c_2,\ldots)$ and $c_i=(A_i,B_i)$.

We will say that a pair $z\in\mathfrak{I}$ and a sequence $s=((A_0,B_0),\ldots,(A_nB_n))\in\mathfrak{S}$ are consistent if $z(0)\bigcap B_n^*=\phi=z(1)\bigcap A_n^*$. Clearly, if s captures z , then s and z are consistent.

Basic open neighborhoods in $\mathbb{F}$ are of the form $O(z,s)$, where $z\in\mathfrak{I}$ and $z\in\mathfrak{S}$ are consistent, an $O(z,s)$ is the set of all $(f,c)\in F$ such that

(A) $f(x)=i\ for\,every\,x\in z(i), i=0,1$;

(B) $s\prec c$.

Note that the condition that s and z are consistent implies that every set $O(z,s)$ is nonempty.

Repeating the proof of Lemma **(**5.3.7), one easily obtains the following

Lemma (5.3.13)[213]: For any consistent $z\in\mathfrak{I}$ and $s\in\mathfrak{S}$, the basic open se $O(z,s)$ in $\mathbb{F}$ contains a neighborhood $O(z,s')$, where s' captures z The proof of Theorem**(**5.3.11) is based on the following auxiliary result.

Lemma (5.3.14)[213]: For any sequence $(X_n)_{n\in\omega}$ of C-sets in $\mathbb{F}$ there exist a sequence $c=((A_n,B_n))_{n\in\omega}\in e'$ And sets $A,B\subseteq\omega$ such that

$-\bigcup_{n\in\omega}A_n^*\subseteq A^*, \bigcup_{n\in\omega}B_n^*\subseteq B^*, A\bigcap B=\phi$,

$-\ \mu(A),\mu(B)\le\frac{1}{6}$;

$-\ for\,any\ n\in\omega,$ the set

$$\left\{f\in C_p(\omega^*,2): f\setminus A^*\equiv 0\, f\setminus B^*\equiv 1\right\}\times\{C\}$$

Is either contained in X_n or disjoint from X_n.

Proof: We inductively define a decreasing sequence $(V_n)_n$ of nonempty open sub- sets of $\mathbb{F}$ and for every n we choose

(i) sequences $\left(\bigcup_k^n\right)_k$ of open sets dense in $(V)_n$ such that $\bigcap_{k\in\omega}U_k^n$ is either contained in X_n or disjoint from X_n;

(ii) $z_n\in\mathfrak{I}$ and $s_n\in\mathfrak{S}$ capturing z_n such that $z_{n-1}\subset z_n, s_{n-1}\prec s_n$, and

$$O(z_n,s_n)\subseteq\bigcap_{i,k\le n}U_k^i.$$

Suppose that the construction has been carried out for $i<n\ (or\,n=0)$. Since

$$X_n\bigcap O(z_{n-1},s_{n-1})$$

is a C−set there is a nonempty open set $V_n \subseteq O(z_{n-1}, s_{n-1})$ and a sequence of itsdense open subsets $(U_k^n)_k$ such that $\bigcap_{k\in\omega} U_k^{\varepsilon}$ is either contained in X_n or disjoint from it. Then the set $G_n = \bigcap_{i,k\le n} U_k^n$ is open and nonempty (because V_i aredecreasing and U_k^i are dense in V_i). Moreover, $G_n \subseteq V_n \subseteq O(z_{n-1}, s_n)$.

Now, we can choose consistent $z_n \in \mathfrak{z}$ and $s_n \in \mathfrak{I}$ such that $z_{n-1} \sqsubset z_n$, $s_{n-1} \prec s_n$, and $O(z_n, s_n) \subseteq G_n$; by Lemma (5.3.13) we can additionally require that s_n captures z_n. The sequence $s_0 < s_1 < \ldots$ defines the unique element $c = ((A_n B_n))_{n\in\omega} \in e'$. We also obtain the sets A, B in the same way as in the proof of Lemma(5.3.9) , applying Lemma (5.3.4)to sequences $(A_n)_n\, and\, (B_n)_n$.

It follows that whenever the function $f \in C_p(\omega^*, 2)$ takes values 0 o A^* and1 on B^*, the pair (f, c) belongs to $\mathbb{F}$ and, for every $n, (f, c) \in O(z_n, s_n)$ since $s_n < c$ and s_n captures z_n. Therefore

$$(f, c) \in \bigcap_{n\in\omega} \bigcap_{i,k\le n} U_k^i = \bigcap_{n\in\omega} \bigcap_{k\in\omega} U_k^n \subseteq \bigcap_{k\in\omega} U_k^n ,$$

for every n, and the lemma follows.

Theorem (5.3.11) can be easily derived from the above lemma. Indeed, if Y_n are C−sets in $(C(\omega^*), \tau_p)$ then $Z_n = Y_n \cap C_p(\omega^*, 2)$ are C−sets in $C_p(\omega^*, 2)$. Let $\pi\colon \mathbb{F} \to C_p(\omega^*, 2)$ be the projection onto the first axis. Then $X_n = \pi^{-1}(Z_n)$ are C−sets in the space $\mathbb{F}$. Applying Lemma (5.3.14) to such sets X_n we conclude that there are two different functions g_1, g_2 with $g_i \backslash A^* \equiv 0, g_i \backslash B^* \equiv 1$ for $i = 1,2$. It follows that (g_i, c) are not separated by X_n and hence g_i are not separated by the sets Y_n.

For a compact space K, we denote by $Ba(C(K), weak), Ba(C(K), \tau_p)$ the Baire σ -algebras in $C(K)$ endowed with the weak topology, or the pointwise topology, respectively.

Theorem (5.3.15)[213]: (Avil´es-Plebanek-Rodr´ıguez).

$$Ba(C(\beta\omega), weak) \neq Ba(C(\beta\omega), \tau_p) .$$

Using results from we can also give a simpler proof of the above theorem:

Proof: We shall show that, for any ultra filter $\wp \in \omega^*$, the measure $\hat{d}_\wp$ is not $Ba(C(\beta\omega), \tau_p)$ -measurable. Assume the contrary. Then there exists a countable subset X of $\beta\omega$ such that $\hat{d}_\wp$ is measurable with respect to the σ-algebra of subsets Of $C(\beta\omega)$ generated by $\{\delta_x : x \in X\}$. Corollary(5.3.5) implies that $\widehat{d}_\wp$ vanishes on the closure of $X \cap \omega^*$ in $\beta\omega$. Take $A \subseteq \omega$ such that $X \cap \omega^* \subseteq \bar{A}$ and $\widehat{d}_\wp(\bar{A}) < 1$. Let $E = \{ f \in C(\beta\omega) : f \backslash \bar{A} \equiv 0\}$. Observe that $\widehat{d}_\wp \backslash E$ is measurable with respect to the σ -algebra generated by $\{\delta_x : x \in X \cap \omega\}$, and for any subset C of $B = \omega \backslash A$ the characteristic function $x_{\bar{C}} : \beta\omega \to \mathbb{R}$ belongs to E . Then the measure $v : p(\omega) \to [0,1]$ defined by

$$v(Z) = d_\wp(Z \cap B) = \widehat{d}_\wp(\overline{Z \cap B}) = \widehat{d}_\wp(x_{\overline{Z\cap B}}),$$

for $Z \in p(\omega)$, is nonzero, Borel-measurable and vanishes on points of ω, a con-tradiction with Proposition (5.3.2).

Note finally that for any compact space K we have the following inclusions

$$\begin{array}{ccccc} & & Ba(C(K), \tau_p) & \subset & Borel(C(k), \tau_p) \\ & & \cap & & \cap \\ Ba(C(K), weak) & \subset & Borel(C(k), weak) & \subset & Borel(C(K), norm) \end{array}$$

The space $K = 2^{\omega_1}$ is an example of a nonmetrizable compactum K for which all the five σ-algebras on $C(k)$ are equal, see [214]. From previous results and the proposition below it follows that all inclusions in the above diagram are strict for the space $K = \beta\omega$. Since $\beta\omega$ is a continuous image of ω^*, this is also the case for $K = \omega^*$. [194].

Proposition(5.3.16)[213]: $Bor\big(C(\beta\omega),\tau_p\big) \nsubseteq. Ba(C(\beta\omega), weak)$.

Proof: Let $\{A_\alpha : \alpha < \omega_1\}$ be a family of almost disjoint subsets of ω . For every $\alpha < \omega_1$ we pick $\mathcal{F}_\alpha \in \omega^*$ such that $A_\alpha \in \mathcal{F}_\alpha$. Let us consider the set $V = \{f \in C(\beta\omega) : f(\mathcal{F}_\alpha) > 0$ for some $\alpha < \omega_1$.

Then V is τ_p-open; we shall check that $V \notin Ba(C(\beta\omega), weak)$.

Suppose otherwise; the V lies in the σ -algebra generated by$\{\delta_n : n \in \omega\}$ and some family $\{\mu_n : n \in \omega\}$, where every μ_n is a probability measure on ω^*. There is $\beta < \omega_1$ such that $\mu_n\big(\overline{A_\beta}\big) = 0$ for every n . Let F be the set of all 0 -1-valued functions in $C(\beta\omega)$ which vanish outside $\overline{A_\beta}$. It follows that the set $F \cap V$ lies in the σ -algebra of subsets of F which is generated by the restrictions of $\mu_n{'s}$ and $\delta_n{'s}$ to F which is simply the σ-algebra generated by δ_n for $n \in A_\beta$. On the other hand, $F \cap V = \{x_{\overline{N}} : N \in \mathcal{F}_\beta, N \subseteq A_\beta\}$, a contradiction, since $\mathcal{F}_\beta \cap 2^{A_\beta}$ is not Borel in the Cantor set 2^{A_β} .

Corollary (5.3.17)[260]: Every basic open set $N(\sigma^2, z^2, s^2)$ in $\mathbb{E}$ contains a neighborhood $N\big(\sigma^2, z^2, {s'}^2\big)$ where ${s'}^2$ captures z^2.

Proof: Indeed, if $(A, A+\epsilon)$ is the final pair in s then for any $\epsilon > 0$, using Lemma(5.3.3) we can find sets $A + 2\epsilon, A + 3\epsilon \subseteq \omega$ of asymptotic densit $y > \epsilon$ and such that $z^2(0) \subseteq (A^* + 2\epsilon) z^2(1) \subseteq (A^* + 3\epsilon), z^2(1) \subseteq (A^* + 3\epsilon)$. Then we can put ${s'}^2 = s^2 \cap (A', A' + \epsilon)$, where $A' = A \cup (A + 2\epsilon), A' + \epsilon = (A + \epsilon) \cup (A + 3\epsilon)$ and ε is small enough.

Corollary (5.3.18)[260]: (Avil´es-Plebanek-Rodr´ıguez).

$$(A+\epsilon)a(C(\beta\omega^2), weak) \neq (A+\epsilon)a\big(C(\beta\omega^2), \tau_p\big) .$$

Using results from we can also give a simpler proof of the above theorem:

Proof: We shall show that, for any ultra filter $\wp \in {\omega^2}^*$, the measure $\hat{d}_\wp$ is not $(A+\epsilon)a\big(C(\beta\omega^2),\tau_p\big)$ -measurable. Assume the contrary. Then there exists a countable subset X of $\beta\omega^2$ such that $\hat{d}_\wp$ is measurable with respect to the σ-algebra of subsets of $C(\beta\omega^2)$ generated by$\{\delta_x : x \in X\}$. Corollary(5.3.5) implies that $\widehat{d}_\wp$ vanishes on the closure of $X \cap {\omega^2}^*$ in $\beta\omega^2$. Take $A \subseteq \omega^2$ such that $X \cap {\omega^2}^* \subseteq \bar{A}$ and $\widehat{d}_\wp(\bar{A}) < 1$. Let $E = \{f \in C(\beta\omega^2) : f \backslash \bar{A} \equiv 0\}$. Observe that $\widehat{d}_\wp \backslash E$ is measurable with respect to the σ -algebra generated by$\{\delta_x : x \in X \cap \omega^2\}$, and for any subset C of $A + \epsilon = \omega^2 \backslash A$ the characteristic function $x_{\bar{c}} : \beta\omega^2 \to \mathbb{R}$ belongs to E . Then the measure $v : p(\omega^2) \to [0,1]$ defined by

$$v(Z) = d_\wp(Z \cap (A+\epsilon)) = \widehat{d}_\wp\big(\overline{Z \cap (A+\epsilon)}\big) = \widehat{d}_\wp(x_{\overline{Z \cap (A+\epsilon)}}),$$

For $Z \in p(\omega^2)$, is nonzero, Borel-measurable and vanishes on points of ω^2, a con-tradiction with Proposition (5.3.2).

Note finally that for any compact space K we have the following inclusions

$$\begin{array}{ccccc} & (A+\epsilon)a\big(C(K),\tau_p\big) & \subset & Borel\big(C(k),\tau_p\big) & \\ & \cap & & \cap & \\ (A+\epsilon)a(C(K), weak) & \subset & Borel(C(k), weak) & \subset & Borel(C(K), norm) \end{array}$$

The space $K = 2^{\omega_1^2}$ is an example of a nonmetrizable compactum K for which all the five σ-algebras on $C(k)$ are equal, see [214]. From previous results and the proposition below it follows that all inclusions in the above diagram are strict for the space $K = \beta\omega^2$. Since $\beta\omega^2$ is a continuous image of ${\omega^2}^*$, this is also the case for $K = {\omega^2}^*$. [194].

Corollary (5.3.19)[260]: $Bor\left(C((\alpha+\epsilon)\omega^2), \tau_p\right) \nsubseteq . Ba(C((\alpha+\epsilon)\omega^2), weak)$.

Proof: Let $\{A_\alpha : \alpha < \omega_1^2\}$ be a family of almost disjoint subsets of ω^2 . For every $\alpha < \omega_1^2$ we pick $\mathcal{F}_\alpha \in {\omega^2}^*$ such that $A_\alpha \in \mathcal{F}_\alpha$. Let us consider the set $V = \{f \in C((\alpha+\epsilon)\omega^2) : f(\mathcal{F}_\alpha) > 0$ for some $\alpha < \omega_1^2$.

Then V is τ_p-open; we shall check that $V \notin Ba(C((\alpha+\epsilon)\omega^2), weak)$.

Suppose otherwise; the V lies in the σ -algebra generated by$\{\delta_n : n \in \omega^2\}$ and some family $\{\mu_n : n \in \omega^2\}$, where every μ_n is a probability measure on ${\omega^2}^*$. There is $\alpha+\epsilon < \omega_1^2$ such that $\mu_n(\overline{A_{\alpha+\epsilon}}) = 0$ for every n . Let F be the set of all 0 -1-valued functions in $C((\alpha+\epsilon)\omega^2)$ which vanish outside $\overline{A_{\alpha+\epsilon}}$. It follows that the set $F \cap V$ lies in the σ -algebra of subsets of F which is generated by the restrictions of μ_n's and δ_n's to F which is simply the σ -algebra generated by δ_n for $n \in A_{\alpha+\epsilon}$. On the other hand, $F \cap V = \{x_{\overline{N}} : N \in \mathcal{F}_{\alpha+\epsilon}, N \subseteq A_{\alpha+\epsilon}\}$, a contradiction, since $\mathcal{F}_{\alpha+\epsilon} \cap 2^{A_{\alpha+\epsilon}}$ is not Borel in the Cantor set $2^{A_{\alpha+\epsilon}}$.

Chapter 6
Factors of Type $\mathrm{II}_1$and Ultra Product II_1 Factors

We study the results rely on the recent work of Ioana, Peterson and Popa, who showd the existence of type II_1 factors without outer automorphisms. Let M_n be a sequence of finite factors with dim $M_n \to \infty$ and denote $M = \mathrm{II}_\omega M_n$ their ultraproduct over a free ultrafilter ω. Some related independence properties for subalgebras in ultraproduct II_1 factors arealso discussed

Section (6.1): Non-Trivial Finite Index Subfactors

We say that a subfactor $\mathrm{N} \subset \mathrm{M}$ of finite index is trivial, if there exists $\mathrm{n} \in \mathbb{N}$ such that $\mathrm{N} \subset \mathrm{M}$ is isomorphic with $1 \otimes \mathrm{N} \subset \mathrm{M_n}(\mathbb{C}) \otimes \mathrm{N}$. We show that there exist type II_1 factors all of whose finite index sub factors are trivial. An M-M-bimodule M HM1is said to be bifinite if dim $(\mathrm{H_M}) < \infty$ and dim (MH) $< \infty$. In the language of Cannes' correspondences, our main theorem then tells that there exist type II_1 factors M such that every bifinite M-M-bimodule is trivial, i.e. isomorphic with a direct sum of copies of $\mathrm{M}^{L^2}(\mathrm{M})_{\mathrm{M}}$.

Such II_1 factors are very special. Indeed, any automorphism $\alpha \in Aut(M)$ gives rise to an M-M-bimodule $H(\alpha)$ on the Hilbert space $L^2(M)$ by the formula

$$x \cdot \xi = a(x)\xi \; and \; \xi \cdot x = \xi x \; for \; all \; x \in M, \xi \in L^2(M) .$$

This M-M-bimodule is trivial if and only if α is an inner automorphism. So, absence of non-trivial finite index subfactors implies absence of outer automorphisms. Further, if p is a projection in M and $\pi : M \to pMp$ a *isomorphism, one considers analogously the M-M-bimodule $p(M)^{L^2}L^2(pM)_M$. Hence, absence of non-trivialfinite index subfactors implies triviality of the fundamental group.

Because of the constructions, the bifinite M-M-bimodules, should be considered as the generalized symmetries of the II_1 factor M. The main statement then becomes that there exist type II_1 factors all of whose generalized symmetries are inner.

In general, computing the outer automorphism group Out (M) of a II_1 factor M is very hard. Connes discovered in [221] that Out (M) is countable whenever M is the group von Neumann algebra of an ICC property (T) group. Only very recently, Ioana, Peterson and Popa showd the existence of type II_1 Factors M with Out (M) trivial, see [223]. Their theorem is an existence result in the same way as is the main result. We comment on that below. Explicit examples of II_1 factors with trivial outer automo rphism group were constructed by Popa and [231], using crossed products by generalized Bernoulli actions and relying on the techniques of Popa's breakthrough von Neumann strong rigidity results in [226], [227]. Note that in [231], it is shown as well that any group of finite presentation can be explicitly realized as the outer autom Orphism group of a II_1 factor.

Also the fundamental group of a II_1factor, introduced by Murray and von Neumann in [118], is very hard to compute, unless, of course, you deal with a McDuff factor and get $\mathbb{R}_+^*$as its fundamental group. Connes showd in [221] that the fundamental group of the group von Neumann algebra of an ICC property (T) group is countable. The first example of a II_1 factor with trivial fundamental group was given by Popa in [228], as the group von Neumann algebra of $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. Many other such examples are given in [223], [226], [227], [231].

In [226], Popa constructs type II_1 factors with an arbitrarily prescribed countable subgroup of $\mathbb{R}_+^*$ as a fundamental group. An alternative construction is given in [223].

The type II_1 factors studied are of the form $M = R \rtimes \Gamma$, where $\Gamma = \Gamma_0 * \Gamma_1$the free product of two infinite groups is and $\Gamma \curvearrowright R$ is an action by outer automorphisms on the hyperfinite II_1 factor R.

We formulate strong conditions on the groups and the actions involved, that ensure that all bifinite M-Mbimodules, are trivial. But, we do not give explicit examples of actions that satisfy all these requirements: as in [223], we rather show the existence of such actions through a Baire category argument.

The following argument, due to Ioana, Peterson and Popa [223] is a key ingredient to show that, under suitable conditions, every bifinite M-M-bimodule is trivial when $M = R \rtimes (\Gamma_0 * \Gamma_1)$. one first assumes that R ⊂ M has the relative property (T). The free product $\Gamma_0 * \Gamma_1$gives rise to a strong deformation property of M.
Combined with the relative property (T) for R ⊂ M, this fixes somehow the position of R inside M. It allows to conclude that any finite index inclusion $\pi : M \rightarrow M^t$ can be unitarily conjugated into one in which $\pi(R) \subset R^t$, see Theorem **(**6.1.6) and Propositions **(**6.1.7) and**(**6.1.8) .

Throughout, $(M, \mathcal{T})$ denotes a von Neumann algebra M with a faithful normal tracial state $\mathcal{T}$. We denote, for all $n \in \mathbb{N}_0$ and all$(M, \mathcal{T})$,

$$M^n: M_n(\mathbb{C}) \otimes M .$$

We use the convention $N_0 = \{1, 2, \dots\}$. If M is a II_1 factor and t > 0, we also introduce the usual notation $M^t = pM^n p$ whenever $p \in M^n$is a projection with non-normalized trace equal to t.

We make an extensive use of Popa's technique of intertwining subalgebras using bimodules. Let $(M, \mathcal{T})$ be a von Neumann algebra with a fixed faithful normal tracial state $\mathcal{T}$. Let A, B ⊂ M be von Neumann subalgebras. We say that A embeds into B inside M and write

$$A \underset{M}{\prec} B$$

If $L^2(M)$ contains a non-zero A-B-subbimodule H that is finitely generated as a right B-module. We write

$$A \overset{f}{\underset{M}{\prec}} B$$

if for every non-zero projection $p \in A' \cap M, L^2(pM)$ contains a non-zero A-B-subbimodule that is finitely generated as a right B-module.
The normalizer of $A \subset$ M consists of the unitaries u ∈ U (M) satisfying $uAu^* = A$ and is denoted by$N_M(A)$. We say that A ⊂ M is regular if $\mathcal{N}_M(A)'' = M$.

If $A \subset (M, \mathcal{T})$ is a von Neumann subalgebra, we say that $\alpha \in M$ quasi-normalizes A if there exist $a_1, \dots, a_n, b_1, \dots, b_m \in M$ satisfying $Aa \subset \sum_{i=1}^n a_i A$ and $aA \subset \sum_{j=1}^m Ab_j$. The set of elements quasinormalizing A is denoted by $QN_M(A)$ and is a unital*j-subalgebra of M containing A. We call quasinormalizer of A inside M the von Neumann algebra $QN_M(A)''$

generated by the elements quasi-normalizingA. If $QN_M(A)'' = M$, we say that the inclusion A ⊂ M is quasi-regular.

If $A \subset (M, \mathcal{T})$ is a von Neumann subalgebra, Jones' basic construction [225] is denoted by $\langle M, e_A \rangle$and definedas the von Neumann algebra acting on $L^2(M)$ generated by A and the orthogonal projection e_A) of $L^2(A)$. Note that A commutes with e_A and that $e_A x e_A = E_A(x) e_A$ for all $x \in M$, where $E_A : M \to A$ denotes the unique t-preserving conditional expectation. Equivalently, $\langle M, e_A \rangle$ equals the commutant of the right A-action on $L^2(M)$.

If (A, t) is a von Neumann algebra with a fixed faithful normal tracial state t and if $H_A$is a right A-module, the commutant $A'$of the right A-action on H is equipped with a canonical normal faithful semifinite trace Tr that can be characterized as follows:

$$Tr(T\,T^*) = \tau(T^*T) \text{whenever } T: L^2(A) \to A: T(\xi a) = (T\,\xi) \text{ a for all } \xi \in H, a \in A.$$

One defines

$$dim(H_A) := Tr(1)$$

and one calls $\dim(H_A)$ the coupling constant or the relative dimension of the right A-module (H_A). As such, the definition of $\dim(H_A)$ depends on the choice of tracial state t on A. Throughout, either A will be a II_1factor, in which case the coupling constant is canonically defined, or A will inherit a trace from a natural ambient II_1factor.

For II_1 factors, the coupling constant is canonically defined and it is then a complete invariant of Hilbert A-modules. If A has a non-trivial center, a complete invariant of Hilbert A-modules can be given in terms of the center-valued trace. We shall only use the following corollary: if $dim(H_A) < \infty$ and $\varepsilon > 0$, there exists a central projection $z \in Z(A), n \in \mathbb{N}$ and a projection $p \in A^N$ such that $t(1 - z) < \varepsilon$ and $(Hz)_A \cong (pL^2(A)^{\otimes n})_A$ as A-modules.

Let $A \subset (M, t)$. Regarding the basic construction and $\langle M, e_A \rangle$ i as the commutant of the right A-action on $L^2(M)$, we get a natural normal faithful semifinite trace Tr on $\langle M, e_A \rangle$. It is characterized by the formula $Tr(xey) = \tau(xy), for\ all\ x, y \in M.$

If ${}_M H_M$ is an M-M-bimodule and A ⊂ M a von Neumann subalgebra, a vector $\xi \in H$ is said to be A-central if $a\xi = \xi a\ for\ all\ a \in A.$

In [228], Popa defined the relative property (T) for an inclusion $A \subset (M, t)$ of von Neumann algebra A into the von Neumann algebra M equipped with a faithful normal tracial state t. An equivalent form of this definition goes as follows. For every $\varepsilon > 0$, there exists a finite subset $\mathcal{F} \subset M$ and a $\delta > 0$ such that every M-M-bimodule that admits a unit vector ξ with the property

$$|\langle \xi, a\xi b \rangle - t(ab)| < \delta\ for\ all\ a, b \in \mathcal{F}$$

admits an A-central vector $\xi_0$satisfying $\|\xi_0 - \xi\| < \varepsilon$

If M is a type II_1factor and ${}_M H_M$ an M-M-bimodule, we say that H is bifinite if dim $({}_M H) < \infty$ and dim $(H_M) < \infty$. The fusion algebra of M is defined as the set of all bifinite M-M-bimodules modulo isomorphism of bimodules and is denoted as FAlg (M). Note that FAlg (M) is equipped with the operations of direct sum and Cannes tensor product, see V.Appendix B in [220] and the brief review below. One has the obvious notion of an irreducible element in FAlg (M), and every element in FAlg(M) is the direct sum of a finite number of irreducibles.

Every M-M-bimodule ${}_MH_M$ has a contragredient M-M-bimodule ${}_M\overline{H}_M$. Its carrier Hilbert space is the adjoint Hilbert space $\overline{H}$ while its bimodule structure is given by

$$x \cdot \bar{\xi} = \overline{\xi a^*} and\ \bar{\xi} \cdot a = \overline{a^* \xi}\ .$$

If H and K are bifinite M-M-bimodules, then H and K are disjoint if and only if H $\otimes_M$K is disjoint from the trivial bimodule ${}_ML^2$ (M)$_M$ if and only if H $\otimes_K$ $\overline{K}$is disjoint from the trivial bimodule.

Finally, recall Frobenius reciprocity: if H, K, L ∈ Flag (M), the multiplicity of H in K $\otimes_M$ L equals the multiplicity of K in H $\otimes_M\overline{L}$ and equals the multiplicity of L in $\overline{K}$ $\otimes_M$H.

We briefly recall the Connes tensor product. If ${}_MH_M$ is an M-M-bimodule, there is a natural dense subbimodule $\mathcal{H} \subset H$ and H is a W^*-M-M-bimodule, meaning that there is an M-valued scalar product on $\mathcal{H}$. More precisely, $\mathcal{H}$ consists of those vectors $\xi \in H$ such that there exists $\lambda > 0$ satisfying $\|\xi a\| \leq \lambda \|a\|_2$

For all a ∈ M. If now ${}_MK_M$ is another M-M-bimodule, the Connes tensor product H$\otimes_M$K is defined as the separation and completion of the algebraic tensor product H $\otimes_{alg}$K for the scalar product

$$\langle a \otimes \xi, b \otimes \eta\rangle := \langle \xi, \langle a, b\rangle_M \eta\rangle$$

The M-M-bimodule structure on H $\otimes_{alg}$K is given by

$$a \cdot (b \otimes \xi) = ab \otimes \xi\ and\ (b \otimes \xi) \cdot a = b \otimes (\xi a)\ .$$

When there is no risk for misunderstanding, the tensor product H $\otimes_M$K is sometimes simply denoted by HK.

In particular, every automorphism $a \in Aut(M)$ defines the element $H(a) \in FAlg(M)$ and as such, one considers Out (M) ⊂ Flag (M).

Note that every bifinite M-M-bimodule is isomorphic with some H(ψ). Moreover, if $\psi : M \to pM^N p$ and $\theta : M \to qM^m q$ are finite index inclusions, the M-M-bimodules H (ψ) and H(θ) are isomorphic if and only if there exists a unitary $u \in p(M_{n,m}(\mathbb{C}) \otimes M)q$ satisfying $\theta(x) = u^*\psi(x)u$ for all x ∈ M. Also note that $H(\psi) \otimes MH(\theta) \cong H((id \otimes \theta)\psi)$.

A subset $\mathcal{F} \subset FAlg(M)$ is called a fusion subalgebra if $\mathcal{F}$ is closed under taking submodules, direct sums and tensor products. An important role is played by freeness between fusion subalgebras.

Definition (6.1.1)[218]: Let M be a II_1factor. Two fusion subalgebras $\mathcal{F}_1, \mathcal{F}_2 \subset FAlg(M)$ are said to be free if thefollowing two conditions hold.

(i) Every tensor product of non-trivial irreducible bimodules, with factors alternatingly from $\mathcal{F}_1$,and $\mathcal{F}_2$is irreducible.

(ii) Two tensor products of non-trivial irreducible bimodules, with factors alternatingly from $\mathcal{F}_1$,and $\mathcal{F}_2$, are equivalent if and only if they are factor by factor equivalent.

Equivalently, $\mathcal{F}_1$,and $\mathcal{F}_2$are free if every tensor product of non-trivial irreducible bimodules, with factors alternatingly from $\mathcal{F}_1$,and $\mathcal{F}_2$, is disjoint from the trivial bimodule.

Whenever a ∈ Aut (M), we defined the bimodule H (a) ∈ Flag (M). So, if $\Gamma \curvearrowright M$ is an outer action, we can regard Γ as a fusion subalgebra of FAlg (M).

Definition (6.1.2)[218]: Let the countable group Γ act outerly on the II_1 factor N. The almost normalizer of $\Gamma \curvearrowright N$ inside FAlg(N) is defined as the fusion subalgebra of FAlg(N) generated by the bifinite N-N-bimodules that can be realized as an N-N-subbimodules of a bifinite $(N \rtimes \Gamma) - (N \rtimes \Gamma)$ −bimodule.

We show some results on the almost normalizing bimodules for $\Gamma \curvearrowright N$ There, the terminology of bimodules almost normalizing $\Gamma \curvearrowright N$, will become more clear as well.

Lemma(6.1.3)[218]: Let $\Gamma \curvearrowright N$ be an outer action on the II_1factor N. If $\Gamma_0 < \Gamma$ is a finite index subgroup, the almost normalizers of $\Gamma_0 \curvearrowright N$ and $\Gamma \curvearrowright N$ inside FAlg(N), coincide.

Proof: Tensoring with the obvious inclusion bimodule

$$H_{incl}(\Gamma_0,\Gamma) = {}_{N\rtimes\Gamma_0} L^2(N \rtimes \Gamma)_{N\rtimes\Gamma}$$

and its contragredient, one goes back and forth between bifinite bimodules for $N \rtimes \Gamma_0$ and $N \rtimes \Gamma$.

We fix infinite groups $\Gamma_0$and Γ_1. We set $\Gamma = \Gamma_0 * \Gamma_1$and take an outer action $\Gamma \curvearrowright N$ of Γ on the II_1factor N. We set $M = N \rtimes \Gamma$, with subalgebras $M_i = N \rtimes \Gamma_i$.

We record from [223] the following result. The first statement follows from [388], and the second one from [223], Theorem (1.1.1).

Theorem(6.1.4)[218]: (Ioana-Peterson-Popa, [223]). The following results hold.

(i) If $Q \subset M$ is a von Neumann subalgebra with the relative property (T), there exists $i \in \{0, 1\}$ such that $Q \underset{M}{\prec} M_i$.

(ii) If t > 0, $i \in \{0, 1\}$ and if $Q \subset M_i^t$ is a von Neumann subalgebra such that $Q \underset{M_i^t}{\not\prec} N^t$, then the quasi -normalizer of Q inside M^t is contained in M_i^t.

Corollary (6.1.5)[218]: Suppose that t > 0 and that $Q \subset M^t$ is a subfactor with the relative property (T) whose quasi-normalizer has finite index in M^tthen $Q \underset{M^t}{\prec} N^t$

Proof: Set $M_i = N \rtimes \Gamma_i$. Replacing Q by $Q^{1/t}$, we may assume that t = 1. Suppose that $Q \underset{M}{\not\prec} N$.. The first statement in (6.1.4) yields $i \in \{0, 1\}$ such that $Q \underset{M}{\prec} M_i$.. Take a projection $\in N^n$, a unital *-homomorphism $\psi : Q \rightarrow pM_i^n p$ and a non-zero partial isometry $v \in (M_{1,n}(\mathbb{C}) \otimes M)p$ satisfying $xv = v\psi(x)$ for all $x \in Q$. By construction, the bimodule

$$\psi(Q)(p(L^2(M_i)^{\oplus n})_{M_i}.$$

is isomorphic with a sub-bimodule of$Q^{L^2}(M)_{M_i}$. Since we are supposing that $Q \underset{M}{\not\prec} N$. We get that $\Psi(Q)_{\underset{pM_i^n p}{\not\prec}}$ Denote by Q_1 the quasi-normalizer of $\psi(Q)$ inside $pM^n p$, The second statement of Theorem(6.1.4) implies that $Q_1 \subset pM_i^n p$. But, if Q_0 denotes the quasi-normalizer of Q inside M, it is clear that $v^* Q_{0v} \subset Q_1$.Since we assume that Q_0 has finite index in M, we arrive at a contradiction.

The following result is a first step towards the main theorem.

Theorem(6.1.6)[218]: Let $\Gamma_0$and $\Gamma_1$be infinite groups, $\Gamma = \Gamma_0 * \Gamma_1$ their free product and $\Gamma \curvearrowright N$ an outer action on the II_1 factor N. Set $M = N \rtimes \Gamma$and suppose that $N \subset M$ has the relative property (T).

If t > 0 and $\pi : M \rightarrow M^t$is a finite index, irreducible inclusion, then

$$\pi(N) \underset{M^t}{\prec} N^t and\ N^t \underset{M^t}{\prec} \pi(N)$$

Proof: By Corollary (6.1.5) , we get that $\pi(N) \underset{M^t}{\prec} N^t$.

Realize $M^t = pM^n p$. Since $\pi(M) \subset M^t$ has finite index, we can take a projection $p_1 \in \pi(M)^m$, a finite index inclusion $\psi : M^t \to p_1\pi(M)^m p_1$ and a non-zero partial is ometry $v \in p(M_{n,m}(\mathbb{C}) \otimes M)p_1$ satisfying $xv = v\psi(x)\, for\, all\, x \in M^t$. Write $\pi(M)^s := p_1\pi(M)^m p_1$. Cutting down if necessary, we may assume that $E_{\pi(M)^s}(v^*v)$ has support p_1.
Then, $\psi(N^t) \subset \pi(M)^s$ has the relative property (T). The quasi-normalizer of $\psi(N^t)$ inside $\pi(M)^s$ contains $\psi(N^t)$ and hence, is of finite index. By Corollary (6.1.5), we get that $\psi(N^t) \underset{\pi(M)^s}{\prec} \pi(N)^s$. so, we find a projection $p_2 \in \pi(N)^k$, a unital*-homomorphism $\theta : \psi(N^t) \to p_2\pi(N)^k p_2$ and a non-zero partial is ometry $w \in p_1(M_{m,k}(\mathbb{C}) \otimes \pi(M))p_2$ satisfying $xw = w\theta(x)\, for\, all\, x \in \psi(N^t)$.
Since $E_{\pi(M)^s}(v^*v)$ has support $p_1$and since w has coefficients in $\pi(M)$, it follows that $vw \neq 0$. Moreover ,$N^t vw \subset vw\pi(N)^k$We have shown that $N^t \underset{M^t}{\prec} \pi(N)$.

First of all, Propositions (6.1.7) and(6.1.8) describe the structure of irreducible bifinite $(P \rtimes \Lambda) - (N \rtimes \Gamma)$ -bimodule containing a bifinite P-N-subbimodule.
The condition of containing a bifinite P-N-subbimodule is of course a very strong one. Typically, an application of the deformation /rigidity techniques explained yields the existence of a P-N-subbimodule of finite N-dimension and the existence of another P-N-subbimodule of finite P-dimension. In Proposition (6.1.9), we show that in good cases this suffices to get the existence of a bifinite P-N-subbimodule.

Note that Proposition(6.1.8) is a generalization of Lemma 8.4 in [223], but we avoid the use of Connes' result about vanishing of 1-cocycles for finite group actions.

Proposition (6.1.7)[218]: Let M_0 be a II_1 factor with regular subfactor N. Suppose that $\Gamma \curvearrowright N$ is an outer action of the ICC group Γ on theII_1 factor N. Let H be an irreducible bifinite $M_0(N \rtimes \Gamma)$ −bimodule containing abifinite $N_0 - N$ subbimodule.
Then, there exists a projection $p \in N^n$ and an irreducible finite index inclusion $\psi : M \to p(N \rtimes \Gamma)^n p$ satisfying
(i) $H \cong H(\psi)\, as\, M_0 - (N \rtimes \Gamma)$-bimodules;
(ii) $\psi(N_0) \subset pN^n p$ and this inclusion has finite index;
(iii)The relative commutant $p(N \rtimes \Gamma)^n p \cap \psi(N_0)' equals\, pN^n p \cap \psi(N_0)'$.

Proof: Let H be an irreducible bifinite $M_0 = (N \rtimes \Gamma)$ -bimodule containing a bifinite $N_0 - N$ subbimodule. Since $N \subset N \rtimes \Gamma$ is irreducible, the von Neumann algebra A consisting of $M_0 - N$-bimodular operators on H is finite-dimensional. Since the elements of A are M -modular, we write A as acting on the right on H.
Take an irreducible bifinite $N_0 - N$ -subbimodule K ⊂ H. Define H as the closed linear span of $M_0\, KA$. We denote by z the orthogonal projection onto H and observe that $z \in Z(A)$. When ever $v \in U(A), Kv \cong K$ as $N_0 - N$-bimodules. So, the regularity of $N_0 \subset M_0$ ensures that H is a direct sum of $N_0 - N$ -bimodules isomorphic with one of the uK for$u \in \mathcal{N}_{M_0}(N)$.

Since Z(A) is a finite-dimensional abelian algebra normalized by the unitaries $u_g, g \in \Gamma$, we can define the finite index subgroup $\Gamma_0 < \Gamma$ consisting of $g \in \Gamma$ such that z and u_g commute. Hence, for $g \in \Gamma_0$, we have $Ku_g \subset \mathcal{H}$, implying that there exists $u \in \mathcal{N}_{M_0}(N_0)$ satisfying $Ku_g \cong uK$ as N-N-bimodules. Next define the subset $I \subset \Gamma$ as
$$I := \{g \in \Gamma \mid Ku_g \cong K\, as\, N_0 - N - bimodules\}.$$

It is easily checked that I is globally normalized by the elements of Γ_0. Moreover, if$g \in I$, we have that $H(\sigma_g)$ is contained in $\overline{K} \otimes_{N_0} K$, implying that I is finite. The ICC property of Γ yields that I = {e}.

Set $M = N \rtimes \Gamma$. Take an irreducible finite index inclusion $\theta : M_0 \to qM^m q$ such that $H \cong H(\theta)$ as $M_0 - M$-bimodules.

The presence of K ⊂ H is then translated to the existence of a non-zero partial isometry $v \in q(M_{m,n}(\mathbb{C}) \otimes M)p_1$ and an irreducible finite index inclusion$\psi_1 : N_0 \longrightarrow p_1 N^n p_1$ such that

$\theta(x)v = v\psi_1(x)\ for\ all\ x \in N_0,$

$K \cong H(\psi_1) as\ N_0 - N - bimodules$

We claim that $p_1 M^n p_1 \cap \psi_1(N_0)' = \mathbb{C}_{p_1}$. Indeed, if $\sum_{g\in\Gamma} x_g u_g$ with $x_g \in p_1 N^n \sigma_g(p_1)$ commutes with $\psi_1(N_0)$, it follows that

X g s g(ψ_1 (y)) = ψ_1 (y)x g for all g ∈ G, y ∈N.

$$x_g \sigma_g(\psi_1(y)) = \psi_1(y) x_g\ for\ all\ g \in \Gamma, y \in N_0$$

So, whenever $x_g \neq 0, Ku_g \cong K \otimes_N H(\sigma_g) \cong K$ and hence g = e. It follows that our relative commutant lives inside $p_1 N^n p_1$and so, is trivial by the irreducibility of $\psi_1(N_0) \subset p_1 N^n p_1$. The claim is shown.

In particular, we conclude that $v^*v = p_1$and that $vv^*1$is a minimal projection in $qM^m q \cap \theta(N_0)'. Also, v^*\theta(N_0)v \subset p_1 N^n p_1$ and this is a finite index inclusion.

Set $B = qM^m q \cap \theta(N_0)'$. By irreducibility of $\theta(M_0) \subset qM^m q$, we know that Ad $\theta(\mathcal{N}_{M_0}(N_0))$ yields an ergodic action on B. Since B admits the minimal projection vv^*, B is finite-dimensional. Denote by z the central support of vv^* in B. Let (f_{ij}) be matrix units forzB with $f_{00} = vv^*$. Take a finite set of $u_K \in \mathcal{N}_{M_0}(N_0)$ such that $\sum_k u_k z u_k^* = q$. Finally, take partial isometries v_{K_i} in N^{n} (enlarging n if necessary) satisfying $v_{k_i} v_{k_i}^* = p_1$ for all k, i and $p = \sum_{k,i} v_{k_i}^* v_{k_i}$a projection in N^{n} Defining

$$w := \sum_{k_i} u_k f_{i0} v v_{ki}\ and\ \psi : M_0 \to p(N \rtimes \Gamma)^n p : \psi(y) = w^* \theta(y) w$$

We are done.

Proposition (6.1.8)[218]: Let $\Lambda \overset{\rho}{\curvearrowright} P$ and $\Gamma \overset{\sigma}{\curvearrowright} N$ be outer actions of the ICC groups Λ, Γ on the II_1 factors P, N.

Suppose that H is a bifinite $(P \rtimes \Lambda) - (N \rtimes \Gamma)$-bimodule containing a bifinite P-N-subbimodule.

Then there exists an irreducible finite index inclusion $\psi : P \rtimes \Lambda \to p(N \rtimes \Gamma)^n p$ with $p \in N^n$ and an isomorphism $\delta : \Lambda \to \Gamma_0$ between finite index subgroups of Λ, Γ, satisfying

(i) $H \cong H(\psi)$,

(ii) $\psi(P) p N^n p$ p and this is a finite index inclusion satisfying $p(N \rtimes \Gamma)^n p \cap \psi(P)' = pN^n p \cap \psi(P)'$;

(iii) for some non-zero projection $z \in Z(pN^n p \cap \psi(P)')$ commuting with $\psi(P \rtimes \Lambda_0$ we have.

$$z\psi(u_g) = \mathrm{x}_{\delta(\mathrm{g})} \mathrm{u}_{\delta(\mathrm{g})} \text{for unitaries } \mathrm{x_s} \in \mathrm{zN^n} \sigma_{\mathrm{s}}(\mathrm{z}) \text{ when } s \in \Gamma_0,$$

Proof: By Proposition (6.1.7), we get $H \cong H(\psi)$ where $\psi : P \rtimes \Lambda \to p(N \rtimes \Gamma)p$ is a finite index inclusion satisfying $p \in N^n, \psi(P) \subset pN^n p$ a finite index inclusion and $p(N \rtimes \Gamma)^n p \cap \psi(P)' = pN^n p \cap \psi(P)'$

Let p_0 be a minimal projection in the finite dimensional algebra $pN^n p \cap \psi(P)'$ and set $\psi_0(x) = \psi(x)p$ for $x \in P$. Define $K = H(\psi_0)$ as a bifinite P-N-bimodule. As in the beginning of the proof of Proposition (6.1.7), we get finite index subgroups $\Lambda_0 < \Lambda$ and $\Gamma_0 < \Gamma$ defined by

$$\Lambda_0 := \{g \in \Lambda \mid \exists h \in G, H(\rho_g)K \cong KH(\sigma_h)\},$$
$$\Gamma_0 := \{h \in \Gamma \mid \exists g \in \Lambda, KH(\sigma_h) \cong H(\rho_g)K\},$$

and an isomorphism $\delta : \Lambda_0 \to \Gamma_0$ such that $H(\rho_g)K \cong KH(\sigma_{\delta(G)})$ for all $g \in \Lambda_0$.

Let $z_0 \in Z(\psi(P)' \cap pN^n p)$ be the central support of p_0 .Take $g \in \Lambda_0$. It follows that $\psi\left(\rho_g(.)\right) z_0 ands\ \sigma_{\delta(g)}(\psi(\cdot)z_0)$ define isomorphic P-N bimodules. So, there exists a unitary $v \in \sigma_{\delta(g)}(z_0)N^n z_0$ such that $v\,\psi(\rho_g(x)) = \sigma_{\delta(g)}(\psi(x))v\ for\ all\ x \in P$. It follows that $u^*_{\delta(g)} d(u_g)$ commutes with $\psi(P)$ and hence, belongs to $pN^n p$. It follows that $z_0\psi(u_g) \in u_{\delta(g)}N^n$ for all $g \in \Lambda_0$. But then.

$$(\psi(uh)^* z_0 \psi(u_h)\psi(u_g) = \left(z_0\psi(u_h)\right)^* (z_0\psi(u_{hg}))$$

Belongs to $u_{\delta(g)}N^n$ as well, for all $h, g \in \Lambda_0$. Setting $z = \bigvee_{h\in\Lambda_0} \psi(u_h)^* z_0 \psi(u_h)$, we are done.

The second condition in the next proposition is quite artificial. In the application, one might as well suppose that A ⊂ M is a quasi-regular inclusion, i.e. $M = QN_M(A)''$. Elsewhere, we plan another application of the proposition: there it is known that whenever $H \subset L^2(M,t)$ is an A-A-subbimodule with $\dim(H_A) < \infty$, then ac tually $H \subset L^2(A)$.

Proposition (6.1.9)[218]: Let (M,τ) (M, t) be avon Neumann algebra with faithful normal tracial state τ . Suppose that A, B ⊂ M are von Neumann subalgebras that satisfy the following conditions.

(i) $A \underset{M}{\prec} B$ and $B \overset{f}{\underset{M}{\prec}} A$

(ii) If $H \subset L^2(M,t)$. is an A-A-subbimodule with dim $(H_A) < \infty$ Then, $H \subset L^2(QN_M\ (A)'')$.

Then there exists a B-A-subbimodule $k \subset L^2(M,\tau)$ satisfying

$$\dim({}_B K) < \infty \text{ and } \dim(K_A) < \infty,$$

So, there exists a projection $p \in M_n(\mathbb{C}) \otimes A$, a non-zero partial isometry $v \in \left(M_{1,n}(\mathbb{C})\otimes M\right)p$ and a unital *-homomorphism $\theta : B \to pA^n p$ Satisfying

$$\theta(B) \subset pA^n p\ \ has\ \text{finite}\ index, and\ bv = v\theta(b)\ for\ all\ b \in B.$$

In the above statement, all dimensions are with respect to the restriction of τ $to\ A$ and B. In particular, the index of $\theta(B) \subset pA^n p$, is defined as $\dim(L^2(pA^n p)B)$, where the right B-module action is through θ.

Proof: Denote by J the anti-unitary operator on $L^2(M,\tau)$ given by $Jx = x^*$. Then, $J(\langle M, e_A\rangle \cap B'J = \langle M, e_B\rangle \cap A'$. So, we get two normal faithful traces on $\langle M, e_A\rangle \cap B'$: one denoted by Tr_A and defined by restricting the trace on $\langle M, e_A\rangle$ and the other denoted by Tr and obtained by applying the previous formula and restricting the trace on $\langle M, e_A\rangle$. Define

$$p = \vee\{p_0 | p_0 0 projection\ in\ \langle M, e_A\rangle \cap B'\ with\ Tr_A\ (p_0) < \infty\},$$
$$q = \vee\{q_0 | q_0 0 projection\ in\ \langle M, e_A\rangle \cap B'\ with\ Tr_B\ (q_0) < \infty\}, \quad (1)$$

It suffices to show that $pq \neq 0$. Indeed, approximating p and q, we get p_0 with $Tr_A(p_0) < \infty$ and q_0 with $Tr_B\ (q_0) < \infty$, satisfying $p_0 q_0 \neq 0$. Taking a spectral projection of the positive operator q, we arrive at an orthogonal projection $r \in \langle M, e_A\rangle \cap B'$ satisfying $Tr_A(r), Tr_B(r) < \infty$. Taking $K = rL^2(M, t)$, thelemma is showd.
Take non-zero partial isometries $v, w \in M_{1,n}(\mathbb{C}) \otimes M$ and, possibly non-unital, *-homomorphisms $\rho : A \to B^n, \theta : B \to A^n$such that

$$av = v\rho(a), bw = w\theta(b)\ for\ all\ a \in A, b \in B.$$

Since $B \underset{M}{\overset{f}{\prec}} A$, we may assume that $v(1 \otimes w) \neq 0$. Note that $ww^* \in M \cap B'$, so that we may assume that $v = v(1 \otimes ww^*)$. By construction, the right A-module generated by the (finitely many) coefficients of $v(1 \otimes w)$, is also a left A-module. Our assumptions imply that the coefficients of $v(1 \otimes w)$ belong to $QN_M(A)''$. With p defined by (1), it is easily checked that $H_0 := pL^2(M, t)$ is a right $QN_M(A)''$-module. By construction,
the coefficients of w belong to H_0 and hence, the coefficients of $v^* = w(v(1 \otimes w))^*$ belong to H as well. By construction, the coefficients of v^*belong to $qL^2(M, t)$ So, we have shown that $pq \neq 0$.
Theorem(6.1.10)[218]: Let Γ_0, Γ_1 be infinite groups acting outerly on the II_1 factor N. Make the following assumptions.
(i) The groups $\Gamma_0, \Gamma_1, \mathbb{Z}$ are two by two not virtually isomorphic.
(ii)The groups $\Gamma_0, \Gamma_1$are not virtually is omorphic to a non-trivial free product.
(iii) Denote by $\mathcal{F}$ the fusion subalgebra of FAlg (N) consisting of the bifinite N-N-bimodules that almost normalize $\Gamma \curvearrowright N$. Then, $\mathcal{F}$ and $\Gamma_1$are free as fusion subalgebras of FAlg (N). (See Definitions (6.1.1) and(6.1.2) for relevant terminology.)
iv) $N \subset N \rtimes \Gamma_0$ 1has the relative property (T).
$Set\ M = N \rtimes (\Gamma_0 * \Gamma_1)$ If $_M H_M$ is a bifinite M-M-bimodule, there exists a finite-dimensional unitary representation $\theta : \Gamma_0 * \Gamma_1 \to U(n)$, such that $_M H_M$ is isomorphic with the M-M-bimodule $H_{rep}\ (\theta)$ defined below.

The M-M-bimodule $H_{rep}\ (\theta)$ is defined as follows. The Hilbert space is given by $\mathbb{C}^n \otimes L^2\ (M)$ and

$$(xu_g) \cdot \xi = (\theta(g) \otimes xu_g)\xi\ and\ \xi \cdot y = \xi(1 \otimes y)$$

For all $\xi \in \mathbb{C}^n \otimes L^2(M), g \in \Gamma_0 * \Gamma_1, x \in N$ and y ∈ M.
A given bifinite M-M-bimodule is of the form H(ψ), where $\psi : N \rtimes \Gamma \to (N \rtimes \Gamma)t$is a finite index inclusion will imply that we may assume that $\psi(N) \subset N^t$ and that the latter is a finite index inclusion. This allows to show Theorems (6.1.11) and (6.1.10). Theorem (6.1.11) follows once we have shown the existence of groups Γ_0, Γ_1 without nontrivial finite-dimensional unitary representations, and actions of these groups on the hyperfinite II_1 factor R satisfying all conditions in Theorem (6.1.10). In order to show this existence, we have to establish the following result: if $\mathcal{F}_1$and $\mathcal{F}_2$are countable fusion subalgebras of FAlg(R), where

R is the hyperfinite II_1 factor, then the set $\alpha \in Aut(R)$ such that $a\mathcal{F}_1 a^{-1}$and $\mathcal{F}_2$ are free, is a Γ -dense subset of Aut(R). This last result generalizes A.3.2 in [223].

Proof: Write and $\Gamma = \Gamma_0 * \Gamma_1 \, and \, M = N \rtimes G$. Let H be a bifinite M-M-bimodal. Combining Theorem**(**6.1.6), Proposition**(**6.1.9) and Proposition **(**6.1.8), we get $H \cong H(\psi)$ where $\psi : M \to pM^n p$ is an irreducible finite index inclusion satisfying

(i) $p \in N^n$ and $\psi(N) \subset pN^n p$ a finite index inclusion,

(ii) $pM^n p \cap \psi(N)' = pN^n p \cap \psi(N)'$,

(iii) $\psi(u_g)z = x_{\delta(g)}u_{\delta(g)}$ for all g $\in \Lambda$, where $\Lambda < \Gamma$ is a finite index subgroup, $\delta : \Lambda \to \Gamma$ an injective homomorphism with finite index image, x_ha unitary in $zN^n\sigma_h(z)$ for all $h \in \delta(\Lambda)$ and z a central projection in $pN^n p \cap \psi(N)'$commuting with $\psi(N \rtimes \Lambda)$.

Denote by K the bifinite N-N-bimodule defined by the inclusion $N \to zN^n z : x : x \to \psi(x)z$. We show that K is a multiple the trivial N-N-bimodule, which will almost end the proof of the theorem.

Set $\Lambda_i := \Gamma_i \cap \Lambda$ and note that Λ_i is a finite index subgroup of Γ_i. We assumed that $\Gamma_0, \Gamma_1, \mathbb{Z}$ have no isomorphic finite index subgroups and that the finite index subgroups of $\Gamma_0, \Gamma_1,$ are freely indecomposable.

Hence, the Kurosh theorem implies that $d(\Lambda_i)$ is a finite index subgroup of $s_i\Gamma_i s_i^{-1}$ for some $s_0 s_1 \in \Gamma$.

Unitary conjugating with u_{s_0} from the beginning, we may assume that $\delta(\Lambda_0)$ is a finite index subgroup of Γ_0 and that $\delta(\Lambda_1)$ is a finite index subgroup of $s\Gamma_1 s^{-1}$. Again unitary conjugating, we may assume that either $s = e \, or \, s \in (\Gamma_1 - \{e\}) \cdots (\Gamma_0 - \{e\})$.

So, the map $N \rtimes \Lambda_0 \to z(N \rtimes \Lambda_0)^n z : y \mapsto \psi(y)z$ defines a bifinite $(N \rtimes \Lambda_0) - (N \rtimes \Lambda_0)$-bimodule that contains the N-N-bimodule K. By Lemma (6.1.3), K is almost normalizing $\Gamma_0 \curvearrowright N$ By our assumptions $K \cup \Gamma_0$ and Γ_1 are free inside FAlg(N). Writing for all $g \in \Lambda_1, d(g) = s\eta(g)s^{-1} for \, \eta(g) \in \Gamma_1$ and s as above, the formula $\psi(u_g)z = x_{\delta(g)}u_{\delta(g)}$implies that $H(\sigma_g)K \cong KH(\sigma_{s\eta(g)s^{-1}})$ for every g $\in \Lambda_1$. Given the form of s, this is acontradiction with the freeness of $K \cup \Gamma_0$ and Γ_1, unless K is a multiple of the trivial N-N-bimodule.

Our claim is shown and we find a non-zero partial is ometry $v \in p(M_{n,1}(\mathbb{C}) \otimes N)$ satisfying

$$\psi(x)v = vx \, for \, all \, x \in N . \qquad (2)$$

Then, $v^*v = 1$ and (2) remains true replacing v by $q\psi(u_g)vu_g^*$ whenever $g \in \Gamma$ and $q \in pN^n p \cap \psi(N)'$. It follows that we can find a unitary $w \in p(M^n, k(\mathbb{C}) \otimes N)$ satisfying $w^*\psi(x)w = 1 \otimes x$ for all $x \in N$. It is now an exercise to check that $w^*\psi(u_g)w = \theta(g) \otimes u_g$ for some representation $\theta : \Gamma \to U(k)$.

Finally, we show the existence of groups and actions satisfying all the requirements in Theorem **(**6.1.10) and moreover such that the groups do not admit finite-dimensional unitary representations.

Theorem (6.1.11)[218]: There exist II_1 factors M with trivial fusion algebra: every bifinite M-M-bimodule is isomorphic with ${}_M(L^2(M)^{\oplus n})_M 1$ for some $n \in \mathbb{N}$.

In particular, M has no outer automorphisms, has trivial fundamental group and only has trivial finite index subfactors: if N $\subset$ M is a finite index subfactor, $(N \subset M) \cong (1 \otimes N \subset$

$M(\mathbb{C}) \otimes N)$ for some $n \in \mathbb{N}$. In particular, every finite index irreducible subfactor of M equals M.

The II_1 factors in the above theorem are of the form $M = R \rtimes \Gamma$, where R is the hyperfinite II_1factor, Γ is the free product of two groups without non-trivial finite dimensional unitary representations and the outer action $\Gamma \curvearrowright N$ satisfies the following specific conditions.

Proof : We have to show that there exist infinite groups Γ_0, Γ together with outer actions on the hyper finite II_1 factor R such that all conditions of Theorem (6.1.10) are satisfied and such that all finite dimensional unitary representations of Γ_0 and Γ are trivial.

Consider the group A_∞ of finite even permutations of$\mathbb{N}$. It is well known that every finite dimensional unitary representation of $A_\infty$is trivial. Consider $\mathbb{Z}/3\mathbb{Z} \subset A_\infty$, identifying 1 and the cyclic permutation of {0, 1, 2}. Finally, consider $\mathbb{Z}/3\mathbb{Z} \subset SL(3,\mathbb{Z})$ identifying 1 and the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ We then define

$$\Gamma_0 = SL(3,\mathbb{Z}) \underset{\mathbb{Z}/3\mathbb{Z}}{*} A_\infty and\ \Gamma_0 = A_\infty$$

As stated above, $\Gamma_1$does not have non-trivial finite dimensional unitary representations. If $\pi : \Gamma_0 \to U(n)$ is a finite dimensional unitary representation,$A_\infty \subset Ker\,\pi$. In particular,. $\mathbb{Z}/3\mathbb{Z} \subset Ker\pi$ Since the smallest normal subgroup of $SL(3,\mathbb{Z})$ containing $\mathbb{Z}/3\mathbb{Z}$, is the whole of $SL(3,\mathbb{Z})$, it follows that $Ker\,\pi = \Gamma_0$.

In particular, Γ_0 and Γ_1 do not have non-trivial finite index subgroups. Both $SL(3,\mathbb{Z})$ and A_∞ are freely indecomposable. Then, the Kurosh theorem implies that Γ_0 is freely indecomposable as well.

We next claim that there exists an outer action of Γ_0 on the hyper finite II_1 factor R such that $R \subset R \rtimes \Gamma_0$ has the relative property (T). First take an outer action of $SL(3,\mathbb{Z})$on R such that $R \subset R \rtimes SL(3,\mathbb{Z})$ has the relative property (T). A way of doing so, goes as follows. Consider the semi-direct product $SL(3,\mathbb{Z}) \ltimes (\mathbb{Z}^3 \times \mathbb{Z}^3)$, where $A \cdot (x,y) = (Ax,(A^{-1})^t y)$ for all $A \in SL(3,\mathbb{Z})$ and $x,y \in \mathbb{Z}^3$. It is clear that $\mathbb{Z}^3 \times \mathbb{Z}^3$ is a subgroup with the relative property (T). Take an $SL(3,\mathbb{Z})$ -invariant non-degenerate 2-cocycle ω on $\mathbb{Z}^3 \times \mathbb{Z}^3$. We then get the required action of $SL(3,\mathbb{Z})\ on\ R = L_\omega(\mathbb{Z}^3 \times \mathbb{Z}^3)$. Next, take any outer action of A_∞ on R. By Cones' uniqueness theorem for outer actions of finite cyclic groups on R (see [222]), we may assume that the actions of $\mathbb{Z}/3\mathbb{Z} \subset A_\infty$ A8 and $\mathbb{Z}/3\mathbb{Z} \subset SL(3,\mathbb{Z})$ coincide. Hence, we get an action of Γ_0 on R. Further modifying the action of A_∞ by applying Proposition (6.1.14), we have shown that there exists an outer action of Γ_0 on R that extends the $SL(3,\mathbb{Z})$ action. Then, $R \subset R \rtimes \Gamma_0$ still has the relative property (T).

Finally, take any outer action of Γ_1 on the hyper finite II_1 factor R. Denote by $\mathcal{F}$ the fusion subalgebra of Flag (R) generated by the bifinite R-R-bimodules almost normalizing $\Gamma_0 \curvearrowright R$. By Lemma (6.1.12) below, $\mathcal{F}$ is countable. It follows from Theorem (6.1.13) below that there exists an automorphism $a \in Aut(R)$ such that $\mathcal{F}$ and $a\Gamma_1 a^{-1}$ are free in the sense of Definition (6.1.1) . Replacing $\Gamma_1 by\ a\Gamma_1 a^{-1}$, all conditions of Theorem(6.1.10) are fulfilled and moreover, Γ only has trivial finite dimensional unitary representations. So, we are done.

Lemma(6.1.12)[218]: Let N be a II_1 factor and $G \curvearrowright N$ an outer action such that $N \subset N \rtimes \Gamma$ has the relative property (T). Then, the almost normalizer of $\Gamma \curvearrowright N$ in FAlg (N) (in the sense of Definition (6.1.2) is countable.

Proof: Set$M = N \rtimes \Gamma$. By contradiction and countability of Γ and $\mathbb{N}$, it is sufficient to show the following: if $n \in \mathbb{N}_0$and if $\psi_i: M \to p_i M^n p_i$ defines an uncountable family of bifinite M-M-bimodules H_i containingnon-zero irreducible bifinite N-N-bimodule $K_i \subset H_i$ there exist $i \neq j$ and $g, h \in \Gamma$ such that $K_i \cong H(\sigma_g) K_j H(\sigma_h),\ as$ N-N-bimodules.
Take $\varepsilon > 0$ and $F \subset M$ finite such that every M-M-bimodule H that admits a vector $\xi \in H$ with the properties $1 - \varepsilon \le \| \xi \| \le 1\ and\ |\langle \xi, a\xi b\rangle - \tau(ab)| < \varepsilon$ for all a, b $\in$ F, actually admits a non-zero N-central vector.

Assume for convenience that $1 \in F$ and consider the ψ_i as non-unital homomorphisms $M \to M^n$. By the pigeon hole principle, we can find $i \neq j$ such that $\|\psi_i(x) - \psi_j(x)\|_2 < \varepsilon \|q_i\|_2$ for all$x \in F$. Consider the M-M-bimodule $p_i L^2(M^n) p_j$ with left action given by ψ_i and right action by ψ_i . The vector $\xi = \|p_i\|_2^{-1} p_i p_j$. satisfies the above conditions and we conclude that $p_i L^2(M^n)_{pj}$ contains a non-zero N-central vector. It follows that there exist irreducible N-N-subbimodules $\tilde{k}_i \subset H_i$ and $\widetilde{K}_j \subset H_j$ with $\widetilde{K}_i \cong \widetilde{K}_j as\ N - N$ bimodules. To conclude to proof, it suffices to observe that for every i, H_i as an N-N-bimodule is a direct sum of irreducible N-N-bimodules isomorphic with $H(\sigma_g) K_j H(\sigma_h), g, h \in \Gamma$.

We show the following crucial result: whenever $\mathcal{F}_1, \mathcal{F}_2$ are countable fusion subalgebras of FAlg(R), where R denotes the hyperfinite II_1 factor, there exists an automorphism $a \in Aut(R)$ such that

$$\mathcal{F}_1^a := H(a-1)\mathcal{F}_1 H(a) \text{ and } \mathcal{F}_2$$

are free. In [219], this implies that any two hyperfinite II_1 nite index subfactors admit a hyperfinite realization of their free composition (see page 94 in [219].

Theorem(6.1.13)[218]: Let R be the hyperfinite II_1factor. Let$\mathcal{F}_1, \mathcal{F}_2$be countable fusion algebras of bifinite R-Rbimodules. Then,

$$\{a \in Aut(R) \mid F_1^a\ and\ \mathcal{F}_2 are\ free\}$$

is a G_δ dense subset of Aut(R).
Recall that if ${}_M H_M$ is an M-M-bimodule and A ⊂ M a von Neumann subalgebra, a vector $\xi \in H$ is said to be A-central if $a\xi = \xi a$ for all a ∈ A. Note that if p denotes the orthogonal projection onto the subspace of A-central vectors, $p\xi$ is precisely the element of minimal norm in the closed convex hull $\overline{co}\{u\xi u^* \mid u \in U(A)\}$.
In what follows, we make use of the following special property for a bifinite bimodules ${}_R H_R$ over the hyperfinite II_1factor R. Fix a free ultrafilter ω on ℕ and consider the ultrapower algebra R^ω. We claim that there exists $n \in \mathbb{N}$ and an R-R-bimodular isometric embedding$V : H \to L^{RL^2}(R^\omega)^{\oplus n}$ in to the n-fold direct sum of $R^{L^2}(R^\omega)_R$ Denoting by $\mathcal{H}$ the W*-bimodule of bounded vectors in H, we can take $V\mathcal{H} \subset M_{n,1}(\mathbb{C}) \otimes R^\omega$. To show the existence of such an embedding, take $\psi : R \to pR^n p$ such that $H \cong H(\psi)$. We can take a partial isometry $A \in M_n(\mathbb{C}) \otimes R^\omega$ satisfying $A^*A = p$ and $(1 \otimes x)A = A\psi(x)$ for all $x \in R$. It then suffices to define

$$p(L^2(R)^{\oplus n}) \to L^2(R^\omega)^{\oplus n}: \xi \mapsto A\xi .$$

Moreover, ${}_R H_R$ does not contain the trivial bimodule if and only if$(id \otimes E_{R' \cap R^\omega})(V\xi) = 0\ for\ all\ \xi \in \mathcal{H}$.

We are now ready to show Theorem (6.1.13) and the proof will be based on the technical Proposition (6.1.17) below.

Proof: Suppose that $H_0, \ldots, H_{2k}$ are irreducible bifinite R-R-bimodules, with H_J non-trivial if $1 \le j \le 2k-1$. When a $\in$ Aut(R) and $H \in FAlg(R)$, we write $H^a := H(a^{-1})HH(a)$ and define

$$K(a) := H_0 H_1^a H_2 H_3^a \cdots H_{2K-1}^a H_{2K}.$$

We have to show that

$$W := \{a \in Aut(R) \mid K(a)\ is\ disjoint\ from\ the\ trivial\ bimodule\}$$

is a G_δ dense subset of Aut (R).

Let $\mathcal{H}_i \subset H_i$ denote the W*-M-M-bimodules that sit densely in side H_i. Take n sufficiently large and take isometric embeddings

$$V_i: H_i \to L^2(R^\omega)^{\oplus n}\ with\ V_i\,\mathcal{H}_i \subset M_{n,1}(\mathbb{C})R^\omega.$$

Denote by $p_{centr}^{K(a)}$ the orthogonal projection onto the R-central vectors of $R^K(a)_R$. When ever $\xi \in \mathcal{H}_i$ and $\varepsilon > 0$, we define

$$W(\xi_0, \ldots, \xi_{2k};\ \varepsilon) := \{a \in Aut(R) \mid \left\| p_{centr}^{K(a)}\ (\xi_0 \otimes \cdots \otimes \xi_{2k} \right\|) < \varepsilon\}.$$

We show three statements.

(i). Every $W(\xi_0, \ldots, \xi_{2k};\ \varepsilon)$ is open in Aut(R).

(ii). Every $W(\xi_0, \ldots, \xi_{2k};\ \varepsilon)$ is dense in Aut(R).

(iii). Taking the intersection of $W(\xi_0, \ldots, \xi_{2k}; \frac{1}{m})$ where m runs through $\mathbb{N}_0$ and the ξ_i run through a countable $\|.\|_2$-dense subset of $\mathcal{H}_i$, we precisely obtain W.

By the Baire category theorem, these statements together show that W is a G
dense subset of Aut(R). To show the first statement, observe that W (ξ $(\xi_0, \ldots, \xi_{2k};\ \varepsilon)$ is the union of all

$$\left\{a \in Aut(R), \left\|\left\|\sum_{i=1}^{n} \lambda_i\, u_i\, u_i^*(\xi_0 \otimes \cdots \otimes \xi_{2k})\right\|\right\|_{k(\alpha)} < \varepsilon\right\}$$

where n runs through $\mathbb{N}_0$. where $\lambda_1, \ldots\ldots, \lambda_n$ runs through all n-tuples of positive real numbers with sum 1 and where $u_1, \ldots, u$ runs through all n-tuples of unitaries in R. All these sets are easily seen to be open.

To show the second statement, set$V_i\,\xi_i = y_i = (y_i(1), \ldots, y_i(n))^t \in M_{n,1}(\mathbb{C}) \otimes R^\omega$. Then, extending an automorphism of R to an automorphism of R^ωin the canonical way, we have.

$$\left\| p_{centr}^{K(a)}\ (\xi_1 \otimes \cdots \otimes \xi_{2k}) \right\|^2 = \sum_{i_0,\ldots,i_{2k}=1}^{n} E_{R'\cap R^\omega} \left\| y_0(i_0)\alpha\big(y_1(i_1)\big)y_2(i_2) \ldots . \alpha\big(y_{2k-1}(k_{2k-1})\big)y_{2k} \right\|_2^2 \quad (3)$$

Fix $\beta \in Aut(R)$. We show that β is in the closure of $W(\xi 0, \ldots, \xi_{2k}; \varepsilon)$ Write R as the infinite tensor product of 2 by 2 matrices, yielding $R = M_{2^s}(\mathbb{C}) \otimes R_s$. It is sufficient to show that, for every $s \in \mathbb{N}$, there exists a unitary u $\in$ Rs such that $(Ad\,u)\beta \in W(\xi_0, \ldots, \xi_{2k};\ \varepsilon)$. The existence of u follows combining (3), and the following observations.

(i) If H_i is disjoint from the trivial bimodule and $\beta \in Aut(R)$ is arbitrary, H_i^{β} does not admit non-zero R-central vectors either and hence, does not even admit non-zero R - central vectors. So,

$$ER_s' \cap R^{\omega}(\beta(y_i(j))) = 0$$

For all j = 1, . . . n, all s and all $\beta \in Aut(R)$.

(ii) By construction, the elements $\beta(y_i(j)) \in R^{\omega}$ quasi-normalize R for all $\beta \in Aut(R)$. Hence, they quasi-normalize R_s for all s.

(iii) We apply Proposition (6.1.17) to the subfactor R_s of the von Neumann algebra generated by R, the $y_{2i}(j)$ *and* $\beta(y_{2i+1}(j))$.

It remains to show the third statement. If $a \in W$, then $a \in W(\xi_0, \ldots, \xi_{2k}; \varepsilon)$ for all $\xi_i$and $\varepsilon > 0$. Conversely, if a belongs to the intersection stated above, we have

$$p_{centr}^{K(a)} \ (\xi_0 \otimes \cdots \otimes \xi_{2k}) = 0$$

for dense families of $\xi_i \in H_i$. But this implies that $p_{centr}^{K(a)} = 0$ and so a ∈ W.

We have the following variant of Theorem (6.1.13) that we use in the proof of Theorem(6.1.11) .

Proposition(6.1.14)[218]: Suppose that the countable groups Γ_0, Γ_1have a common finite subgroup K. Let $\Gamma_0 \overset{*}{_k} \Gamma_1$ act on the hyperfinite II_1 factor R. Suppose that both Γ_0 and $\Gamma_1$for act outerly. Denote by $Aut_K(R)$ the automorphisms of R that commute with all the automorphisms in K. Then,

$\{a \in Aut_K(R)$ | The subgroups $\Gamma_0$and $a\Gamma_1 a^{-1}$ of Out(R) are free with amalgamation over K} is $a\ G_\delta$ dense subset of $Aut_K(R)$.

Proof: One can almost entirely copy the proof of Theorem(6.1.13) , using the following observation. Let a ∈ Aut(R) be such that $\sigma_k\alpha$ is outer for every k ∈ K. Denote by R^K the fixed point algebra of K. We claim that the R-R-bimodule H (a) has no non-zero RK⊂ R implies that there exists a unitary $v \in R$ such that $va(x)v^* = x$- for all $x \in R^K$. By Jones' uniqueness theorem for outer actions of finite groups (see [224]), we may assume that the action of K is dual and conclude that $(Ad\ v)\alpha = \sigma_k$ for some $k \in K$. This contradicts our assumption and shows that $H(a)$ has no non-zero R^K-central vectors. Writing R^K as an infinite tensor product of 2 by 2 matrices, we get $RR^K = M_{2^K}(\mathbb{C}) \otimes R_K$.

If $A \in R^{\omega}$ is a unitary implementing a, it follows as in the proof of(6.1.13) that $E_{R_K'} \cap R^{\omega}(A) = 0$. this is again the starting point to apply Proposition(6.1.17) .

The following is the crucial result to obtain Theorem(6.1.13) . Most of the proof is taken almost literally from Lemmas 1.2,1.3 and 1.4 in [260]. We repeat the argument, since slight modifications are needed: in [230], the relative commutant $N' \cap M$ is assumed to be finite-dimensional, while we assume that N is a factor and the inclusion N ⊂ M quasi-regular. This forces us to show the extra below.

Lemma(6.1.15)[218]: Let (M, t) be a von Neumann algebra with faithful normal tracial state t. Let N ⊂ M be a von Neumann subalgebra. Suppose that N is a factor of type II_1 and that N is quasi-regular in M. Let $f \in N$ be a non-zero projection and $V \subset M$ a finite subset such that $E_{N' \cap M}(fAf) = 0$ for all $A \in V$.

For every $\varepsilon > 0$ and every $K \in \mathbb{N}_0$, there exists a partial isometry $v \in fNf$ satisfying $vv^* = v^*v, t(vv^*) \geq t(f)/4$ and

$$\left\|E_{N'\cap M}(A_0 v^{k_1} A_1 v^{k_2} A_2 \cdots v^{k_n} A_n\right\|_2 < \varepsilon$$

for all $1 \le n \le K, 1 \le |k_i| \le K, A_0, A_n \in V \cup \{1\}$ and $A_1, \ldots, A_{n-1} \in V$.
Here, and in what follows, we use the convention that $v_0 := vv^*$and $v^{-k} := (v^*) for\ k \in \mathbb{N}_0$, whenever v isa partial isometry satisfying $vv^* = v^*v$.

Proof: We may assume that $\|A\| \le 1$ for all $A \in V$. Since $\|z\|_2^2 \le \|z\| \, \|z\|_1$, we show the following: for every $\varepsilon > 0$ and every $K \in \mathbb{N}_0$, there exists a partial isometry $v \in fNf$ such that $vv^* = v^*v, t(vv^*) \ge t(f)/4$ and

$$\left\|E_{N'\cap M}(A_0 v^{k_1} A_1 v^{k_2} A_2 \cdots v^{k_n} A_n\right\|_1 < \varepsilon$$

for all $1 \le n \le K, 1 \le |k_i| \le K, A_0, A_n \in V \cup \{1\}$ and $A_1 \cdots v_{n-1} \in V$.
Fix $\varepsilon > 0$ and $K \in \mathbb{N}_0. Let\ \varepsilon_0 > 0$ and define $\varepsilon_n = 2^{n+1}\varepsilon_{n-1}, up\ to\ \varepsilon_k$ take ε_0 small enough such that $\varepsilon_K < \varepsilon$. Define I as the set of partial isometries $v \in fNf$ satisfying $vv^* = v^*v$ and

$$\left\|E_{N'\cap M}(A_0 v^{k_1} A_1 v^{k_2} A_2 \cdots v^{k_n} A_n\right\|_1 \le \varepsilon_n t(vv^*)$$

For all $1 \le n \le K, 1 \le |k_i| \le K, A_1, \ldots, A_{n-1} \in V, A_0 \in V \cup fV \cup \{1\}\ and\ A_n \in V \cup Vf \cup \{1\}$.
Order I by inclusion of partial isometries. By Zorn's lemma, take a maximal element $v \in I$ and set $p = vv^*$. It might be that $v = 0$. If $\tau(p) \ge \tau(f)/4$, we are done. Otherwise $t(p) < t(f)/4$ and we set$p_1 := f - p$. Note that $t(p)/t(p_1) = 1/3$. Write $M_1 := p_1 M p_1$, with normalized tracial state t_1 and corresponding norms $\|\cdot\|_{1,M_1}\ and\ \|\cdot\|_{2,M_1}$. Applying Theorem A.1.4 in [229] to the inclusion $p_1 N p_1 \subset p_1 M p_1$, take a non-zero projection $q \in p_1 N p_1$ such that

$$\left\|qxq - E_{(N'\cap M)_{p_1}}(p_1 x p_1)q\right\|_{2,M_1} \le \varepsilon_0 \|q\|_{2,M_1}$$

for all $x = A_1 v^{k_1} \cdots v^{k_{s-1}}, A_s$ and all $1 \le s \le K, 1 \le |k_i| \le K$ and $A_1, \ldots, A_s \in V$. We shall show that aunitary $w \in qNq$ can be chosen in such a way that $v + w \in I$. This then contradicts the maximality of v.
Let $x = A_1 v^{k_1} \cdots v^{k_{s-1}} A_s$with $1 \le s \le K, 1 \le |k_i| = K$ and $A_1, \ldots, A_s \in V$. Observe that

$$\left\|qxq - E_{(N'\cap M)P_1}(p_1 x p_1)q\right\|_{1,M_1} \le \left\|qxq - E_{(N'\cap M)P_1}(p_1 x p_1)q\right\|_{2,M_1} \|q\|_{2,M_1} \le \varepsilon_0 t_1(q)$$

One checks that$\left\| E_{(N'\cap M)P_1}(p_1 x p_1)q\right\|_{1,M_1} \le \left\| E_{(N'\cap M)P_1}(x p_1)\right\|_1 \tau_1(q)/\tau(p_1)$. On the other hand,

$$\left\| E_{(N'\cap M)P_1}(x p_1)\right\|_1 \left\|E_{(N'\cap M)P_1}(xf)\right\|_1 + \left\|E_{(N'\cap M)}(xp)\right\|_1$$
$$= \left\|E_{(N'\cap M)}(xf)\right\|_1 \left\|E_{(N'\cap M)P_1}(vxv^*) \le (\varepsilon_{s-1} + \varepsilon_{s+1})t(p)\right\|_1.$$

It follows that $\left\| E_{(N'\cap M)P_1}(p_1 x p_1)q\right\|_{1,M_1} \le \tau_1(q)(\varepsilon_{s-1} + \varepsilon_{s+1})/3$ Altogether, we conclude that

$$\|qxq\|_1 \le \frac{\varepsilon_{s+1}\tau(q)}{2}. \qquad (4)$$

By Lemma (6.1.16) below, take a unitary $w \in qNq$ such that

$$\left\|E_{N'\cap M}\left(A_0 v^{k_1} \ldots A_{j-1} w^{k_j} A_j \ldots v^{k_n} A_n\right)\right\|_1 \le \frac{\varepsilon_n \tau(q)}{4n}$$

For all $1 \leq n \leq K, 1 \leq j \leq n, 1 \leq |k_i| \leq K, A_1, \ldots, A_{n-1} \in V \cup Vf \cup \{1\}. and\ A_n \in V \cup Vf \cup \{1\}$

Claim: the partial isometry $v + w$ belongs to I, contradicting the maximality of v. To show the claim, take $1 \leq n \leq K, 1 \leq |k_i| \leq K, A_1, \ldots, A_{n-1} \in V, A_0 \in V \cup fV \cup \{1\}$ and $A_n \in V \cup Vf \cup \{1\}$. We develop the sums in the expression

$$E_{N'\cap M}(A_0(v^{k_1} + w^{k_1})A_1(v^{k_2} + w^{k_2})A_2 \ldots (v^{k_2} + w^{k_2})A_n). \quad (5)$$

(i) There is one term with only $v's$ appearing. Its $\|\cdot\|_1$ -norm is bounded by $\varepsilon_n t(p)$, because$v \in I$.

(ii) There are n terms with w appearing at one place. Each term has it s $\|\cdot\|_1$. bounded by $\frac{\varepsilon_n t(p)}{4n}$ Altogether, their $\|\cdot\|_1$-norm is bounded by ent(q)/4.

(iii) There is 1 term with w appearing in position 1 and position n and with v's in the other positions. This term contains the subexpression

$$qA_1 v^{k_2} \cdots v^{k_{n-1}} A_{n-1} q \ .$$

Because of (4), thek$\|\cdot\|_1$-norm of this term is bounded by $\varepsilon_n t(q)/2$.

(iii) There are less than 2^n-1n terms where w appears on at least two positions that are not exactly the positions 1, n. In every such term, we have the subexpression

$$qA_i v^{k_{i+1}} \cdots v^{k_j} A_j q \ .$$

with $1 \leq i \leq j \leq n - 1\ and\ 0 \leq j - i \leq n - 3. By$ (4) , the $\|\cdot\|_1$ -norm of this subexpression is bounded by $\varepsilon_{n-1}\ t(q)/2$. It follows that the sum of all the terms of this type has $\|\cdot\|_1$-norm bounded by $2^{n-1}\varepsilon_{n-1}\ t(q) \leq = \varepsilon_n t(q)/4$

It follows that the $\|\cdot\|_1$.-norm of the expression in (5) is bounded by $\varepsilon_n(\tau(p) + \tau(q)) = \varepsilon_n t(p + q)$, proving that $v + w \in I$.

Lemma(6.1.16)[218]: Let (M, t) be a von Neumann algebra with faithful normal tracial state. Let N ⊂ M be a von Neumann subalgebra. Suppose that N is a factor of type II_1 and that N is quasi-regular in M. If w is a bounded sequence in N that converges weakly to 0, then

$$\|E_{N'\cap M}(aw_n b)\|_2 \to 0$$

for all a, b ∈ M.

Proof: Step 1. Let $a \in M$ with $\|a\| \leq 1$. The sequence $\|E_{N'\cap M}(aw_n)\|_2$ converges to 0, whenever w is a bounded sequence in N that converges weakly to zero. Indeed, writing$E_{N'\cap M} = E_{N'\cap M} o E_{N\vee(N'\cap M)}$, we may assume that $a \in N \vee (N' \cap M)$. So, we may assume that $a = xy$ with $x \in N' \cap M$ and $y \in N$. Because N is a factor, $E_{N'\cap M}(z) = t(z)1$ for all $z \in N$. Hence, $E_{N'\cap M}(yw_n)x = t(yw_n)x$ and this last sequence converges to 0 in $\|\cdot\|_2$.

Step 2. Let $\xi \in L^2(M)$. The sequence $\|E_{N'\cap M}(\xi w_n)\|_2$converges to 0, whenever w_n is a bounded sequence in N that converges weakly to zero. This follows immediately from Step 1.

Step 3, proof of the lemma. Define K as the closure of NbN in $L^2(M)$. Since $N \subset M$ is quasiregular, we may assume that $\dim(K_N) < \infty$ We then find $\xi \in M_{1,n}(\mathbb{C}) \otimes K$, and a, possibly non-until, *-homomorphism $\psi : N \to M_n(\mathbb{C}) \otimes N$ .such that $x\xi = \xi\psi(x)$ for all $x \in N$ and such that K equals the closure of $\xi(M_{n,1}(C) \otimes N)$. So, we may assume that $b = \xi d$ for some $d \in M_{n,1}(\mathbb{C}) \otimes N$. But then, $aw_n b = a\xi\psi(w_n)d$.

Since $\psi(w_n)$is a bounded sequence in $(M_{n,1}(C) \otimes N)$ that converges weakly to zero, the lemma follows from Step 2.

Proposition (6.1.17)[218]: Let (M, t) be a von Neumann algebra with faithful normal tracial state t. Let N ⊂ M be a von Neumann subalgebra. Suppose that N is a factor of type II_1 and that N is quasi-regular in M. Let $V \subset M$ be a finite subset such that $E_{N'\cap M}(A) = 0$ for all A ∈ V.

For every $\varepsilon > 0$ and every $K \in \mathbb{N}_0$, there exists a unitary $u \in N$ such that

$$\left\|E_{N'\cap M}(A_0 u^{k1} A_1 u^{k2} A_2 \cdots u^{kn} A_n)\right\|_2 < \varepsilon$$

for all $1 \leq n \leq K, 1 \leq |k_i| \leq K, A_0, A_n \in V \cup \{1\}$ and $A_1, \ldots, A_{n-1} \in V$.

Proposition (6.1.17) is shown below, after the following preliminary result.

Proof: Let $N \subset (M, t)$ be a quasi-regular inclusion. Suppose that N is a II_1 factor.

Claim 1. Let ω be a free ultrafilter on ℕ and $f \in N^\omega$ a non-zero projection. If $V \subset N^\omega$ is a countable set with $E_{(N'\cap M)^\omega}(fxf) = 0$ for all $x \in V$, there exists a non-zero partial isometry $v \in fN^\omega f$ satisfying $vv^* = v^*v$ and $E_{(N'\cap M)^\omega}(y) = 0$ for every product y with factors alternatingly from V and $\{v^k | k \in \mathbb{Z}, k \neq 0\}$.

Proof : Write $f = (f_n)$ where f_n a non-zero projection in N for every n. Write $V = \{x|k \in \mathbb{N}\}$ and choose representatives $x_k = (x_k, n)_n$ such that $E_{N'\cap M}(f_n x_{k,n} f_n) = 0$ for all k, n. By Lemma **(**6.1.15), take partial isometries $v_n \in f_n N f_n$ such that $v_n v_n^* = v_n^* v_n, t(v_n v_n^*) \geq t(f_n)/4$ *and* $\|E_{N'\cap M}(y)\|_2 < 1/n$ whenever y is a product of at most $2n + 1$ factors alternatingly from $\{x_{0,n}, \ldots, x_{n,n}\}$ *and* $\{v_n^k | 1 \leq |k| \leq n\}$.

Then, $v := (v_n)$ does the job.

Claim 2. Let ω be a free ultrafilter on ℕ and $V \subset M^\omega$ a countable set with $E_{(N'\cap M)^\omega}(x) = 0$ for all $x \in V$. There exists a unitary $u \in N^\omega$ satisfying $E_{(N'\cap M)^\omega}(y) = 0$ for every product y with factors alternatingly from V and $\{u^K | k \in \mathbb{Z}, k \neq 0\}$.

Proof: Define I as the set of partial isometries $v \in N^\omega$ satisfying $vv^* = v^*v$ and $E_{(N'\cap M)^\omega}(y) = 0$whenever y is a product with factors alternatingly from V and $\{v^k | k \in \mathbb{Z}, k \neq 0\}$. By Zorn's lemma, I admits a maximal element v. If v is a unitary, we are done. Otherwise, $vv^* = p < 1$ and we set $f = 1 - p$.

Define W as the (countable) set of products y with factors alternatingly from V and $\{v^k | k \in \mathbb{Z}, k \neq 0\}$ and such that the product y starts and ends with a factor from V. Observe that $E_{(N'\cap M)^\omega}(fyf) = 0$ for all $y \in W$. Indeed,

$$E_{(N'\cap M)^\omega}(fyf) = E_{(N'\cap M)^\omega}(y) - E_{(N'\cap M)^\omega}(yp) = 0 - E_{(N'\cap M)^\omega}(vyv^*) = 0; .$$

Using Claim 1, take a non-zero partial isometry $w \in fN^\omega f$ satisfying $ww^* = w\,w^*$ and $E_{(N'\cap M)^\omega}(y) = 0$for every product y with factors alternatingly from W and $\{w^k | k \in \mathbb{Z}, k \neq 0\}$. Then, $v + w \in I$, contradicting
the maximality of v.

Proof: Consider $V \subset M \subset M^\omega$ with $E_{(N'\cap M)}(x) = 0$ for all $x \in V$. Claim 2 yields a unitary $u \in N^\omega$ such that $E_{(N'\cap M)^\omega}(y) = 0$ for every product y with factors alternatingly from V and$\{u^k | k \in \mathbb{Z}, k \neq 0\}$. Writing $u = (u_n)$ with ununitary for all n, some u_n for n big enough will do the job since the elements of V are represented by constant sequences in M^ω.

We briefly recall Popa's technique of intertwining subalgebras of a II_1 factor using bimodules, introduced in [226], [228] (see also Appendix C in [232]).

Definition (6.1.18)[218]: (6.1.21). Let (M, t) be a von Neumann algebra with faithful normal tracial state$t$. Suppose that A, B ⊂ M are von Neumann subalgebras. We say that A embeds into B inside M and write $A \underset{M}{\prec} B$, if one of the following equivalent conditions is satisfied.
(i) $L^2 < (M, t)$ admits a non-zero A-B-subbimodule H satisfying $\dim(H_B) < \infty$
(ii) $\langle M, e_B\rangle + \cap A'$contains an element x with $0 < Tr(x) < \infty$.
(iii) There exists a projection $p \in B^n$, a normal*B-homomorphism $\psi : A \to pB^np$ and a non-zero partialisometry $v \in M_{1,n}(\mathbb{C}) \otimes M$ satisfying $xv = v\psi(x)$ for all $x \in A$.
(iv) There does not exist a generalized sequence $(u_i)_{i\in I}$of unitaries in A satisfying

$$\|E_B(au_ib)\|_2 \to 0 for\ all\ a, b \in M .$$

We write $A \underset{M}{\overset{f}{\prec}} B$ B, if one of the following equivalent conditions is satisfied.

(vi) For every non-zero projection $p \in M^nA', L^2(pM, t)$ admits a non-zero Ap-B-subbimodule H satisfying $\dim(H_B) < \infty$.
vii) For every $\varepsilon > 0$, there exists a projection $p \in B^n$, a normal* -homomorphism $\psi : A \to pB^np$ and a partial isometry $v \in M_{1,n}(\mathbb{C}) \otimes M$ satisfying $\tau(1 - vv^*) < \varepsilon$ and $xv = v\psi(x)$ for all $x \in A$.
Let $A \subset (M, t)$. The set $QN_M(A)$ of elements quasi-normalizing A was introduced, as well as the quasi-normalizer $QN_M(A)''$. Then, $QN_M(A)''$is as well the weak closure of all $x \in M$ for which the closure of AxA in $L^2(M, t)$ has finite dimension both as a right A-module and as a left A-module.
Let $A, B \subset (M, t)$. Define $p = \vee\{p_0 | p_0 \in \langle M, e_E\rangle \cap A'$ is a projection satisfying $Tr(p_0) < \infty\}$.

Then, $pL^2(M, t)$ clearly is an A-B-subbimodule of $L^2(M, t)$. In fact, it is easy to check that it actually is a $QN_M(A)'' - QN_M(B)''$-subbimodule.

Corollary (6.1.19)[260]: Suppose that $\epsilon \geq 0$ and that $Q_{r-2} \subset M_{r-2}^{1+\epsilon}$ is a subfactor with the relative property (T) whose quasi-normalizer has finite index in $M_{r-2}^{1+\epsilon}$ then Q $\underset{M_{r-2}^{1+\epsilon}}{\prec} N_{r-2}^{1+\epsilon}$.

Proof: Set $(M_{r-2})_i = N_{r-2} \rtimes (\Gamma_{r-2})_i$. Replacing Q_{r-2} by $Q_{r-2}{}^{\frac{1}{1+\epsilon}}$, we may assume that $\epsilon = 0$. Suppose that $Q_{r-2} \underset{M_{r-2}}{\not\prec} N_{r-2}$. The first statement yields $i \in \{0,1\}$ such that $Q_{r-2} \underset{M_{r-2}}{\prec} (M_{r-2})_i$.. Take a projection $\in N_{r-2}^n$, a unital *-homomorphism $\psi : Q_{r-2} \to p(M_{r-2})_i^n p$ and a non-zero partial isometry $v \in ((M_{r-2})_{1,n}(\mathbb{C}) \otimes M_{r-2})p$ satisfying $xv = v\psi(x)$ for all $x \in Q_{r-2}$. By construction, the bimodule

$$\psi(Q_{r-2})(p(L^2((M_{r-2})_i)^{\oplus n})_{(M_{r-2})_i}.$$

is isomorphic with a sub-bimodule of $Q_{r-2}^{L^2}(M_{r-2})_{(M_{r-2})_i}$. Since we are supposing that $Q_{r-2} \underset{M_{r-2}}{\not\prec} N$. We get that $\Psi(M_{r-2}) \underset{pM_i^n p}{\not\prec}$ Denote by Q_r the quasi-normalizer of $\psi(Q_{r-2})$ inside $pM_{r-2}^n p$, The second statement of Theorem**(**6.1.4) implies that $Q_r \subset p(M_{r-2})_i^n p$. But, if Q_{r-1} denotes the quasi-normalizer of Q_{r-2} inside M_{r-2}, it is clear that $v^*Q_{r-1} \subset Q_r$. Since we assume that Q_{r-1} has finite index in M_{r-2}, we arrive at a contradiction.

Corollary (6.1.20)[260]: Let Γ_{r-1} and Γ_r be infinite groups, $\Gamma_{r-2} = \Gamma_{r-1} * \Gamma_r$ their free product and $\Gamma_{r-2} \curvearrowright N_{r-2}$ an outer action on the II_1 factor N_{r-2}. Set $M_{r-2} = N_{r-2} \rtimes \Gamma_{r-2}$ and suppose that $N_{r-2} \subset M_{r-2}$ has the relative property (T).
If $\epsilon \geq 0$ and $\pi : M_{r-2} \to M_{r-2}^{1+\epsilon}$ is a finite index, irreducible inclusion, then

$$\pi(N) \prec_{M_{r-2}^{1+\epsilon}} N_{r-2}^{1+\epsilon} \text{ and } N_{r-2}^{1+\epsilon} \prec_{M_{r-2}^{1+\epsilon}} \pi(N_{r-2})$$

Proof: By Corollary (6.1.5) , we get that $\pi(N_{r-2}) \prec_{M_{r-2}^{1+\epsilon}} N_{r-2}^{1+\epsilon}$.
Realize $M_{r-2}^{1+\epsilon} = pM_{r-2}^{n}p$. Since $\pi(M_{r-2}) \subset M_{r-2}^{1+\epsilon}$ has finite index, we can take a projection $p_1 \in \pi(M_{r-2})^m$, a finite index inclusion $\psi : M_{r-2}^{1+\epsilon} \to p_1\pi(M_{r-2})^m p_1$ and a non-zero partial is ometry $v \in p((M_{r-2})_{n,m}(\mathbb{C}) \otimes M_{r-2})p_1$ satisfying $xv = v\psi(x)$ for all $x \in M_{r-2}^{1+\epsilon}$. Write $\pi(M_{r-2})^s := p_1\pi(M_{r-2})^m p_1$. Cutting down if necessary, we may assume that $E_{\pi(M_{r-2})^s}(v^*v)$ has support p_1.
Then, $\psi(N_{r-2}^{1+\epsilon}) \subset \pi(M_{r-2})^s$ has the relative property (T). The quasi-normalizer of $\psi(N_{r-2}^{1+\epsilon})$ inside $\pi(M_{r-2})^s$ contains $\psi(N_{r-2}^{1+\epsilon})$and hence, is of finite index. By Corollary (6.1.5), we get that $\psi(N_{r-2}^{1+\epsilon}) \prec_{\pi(M_{r-2})^s} \pi(N_{r-2})^s$. so, we find a projection $p_2 \in \pi(N_{r-2})^k$, a unital*-homomorphism $\theta : \psi(N_{r-2}^{1+\epsilon}) \to p_2\pi(N_{r-2})^k p_2$ and a non-zero partial is ometry $w \in p_1((M_{r-2})_{m,k}(\mathbb{C}) \otimes \pi(M_{r-2}))p_2$ satisfying $xw = w\theta(x)$ for all $x \in \psi(N_{r-2}^{1+\epsilon})$.
Since $E_{\pi(M_{r-2})^s}(v^*v)$ has support $p_1$and since w has coefficients in $\pi(M_{r-2})$, it follows that $vw \neq 0$. Moreover ,$N_{r-2}^{1+\epsilon}vw \subset vw\pi(N_{r-2})^k$. We have shown that $N_{r-2}^{1+\epsilon} \prec_{M_{r-2}^{1+\epsilon}} \pi(N_{r-2})$.

Corollary (6.1.21)[260]: Let (M, t) be a von Neumann algebra with faithful normal tracial state. Let N ⊂ M be a von Neumann subalgebra. Suppose that N is a factor of type II_1 and that N is quasi-regular in M. If w^r is a bounded sequence in N that converges weakly to 0, then

$$\|E_{N'\cap M}(aw_n^r(a+\epsilon))\|_2 \to 0$$

for all $a, (a+\epsilon) \in$ M.
Proof: Step 1. Let $a \in M$ with $\|a\| \leq 1$. The sequence $\|E_{N'\cap M}(aw_n^r)\|_2$ converges to 0, whenever w^r is a bounded sequence in N that converges weakly to zero. Indeed, writing$E_{N'\cap M} = E_{N'\cap M} o E_{N\vee(N'\cap M)}$, we may assume that $a \in N \vee (N' \cap M)$. So, we may assume that $a = x(x+\epsilon)$ with $x \in N' \cap M$ and $(x+\epsilon) \in N$. Because N is a factor, $E_{N'\cap M}(x+2\epsilon) = t(x+2\epsilon)$ for all $(x+2\epsilon) \in N$. Hence, $E_{N'\cap M}((x+\epsilon)w_n^r)x = t((x+\epsilon)w_n^r)x$ and this last sequence converges to 0 in $\|\cdot\|_2$.
Step 2. Let $\xi \in L^2(M)$. The sequence $\|E_{N'\cap M}(\xi w_n^r)\|_2$converges to 0, whenever w_n^r is a bounded sequence in N that converges weakly to zero. This follows immediately from Step 1.
Step 3, proof of the lemma. Define K as the closure of $N(a+\epsilon)N$ in $L^2(M)$. Since $N \subset M$ is quasiregular, we may assume that $\dim(K_N) < \infty$. We then find $\xi \in M_{1,n}(\mathbb{C}) \otimes K$, and a, possibly non-until, *-homomorphism $\psi : N \to M_n(\mathbb{C}) \otimes N$. Such that $x\xi = \xi\psi(x)$ for all $x \in N$ and such that K equals the closure of $\xi(M_{n,1}(C) \otimes N)$. So, we may assume that $a + \epsilon = \xi d$ for some $d \in M_{n,1}(\mathbb{C}) \otimes N$. But then, $aw_n^r(a+\epsilon) = a\xi\psi(w_n^r)d$.
Since $\psi(w_n^r)$is a bounded sequence in $(M_{n,1}(C) \otimes N)$ that converges weakly to zero, the lemma follows from Step 2.

Section (6.2): Independence Properties in Subalgebras

We continue the investigation of independence properties in subalgebras of ultraproduct II_1 factors, from [229], [256]. The main result we show along these lines is the following:

Theorem (6.2.1)[233]: Let M_n be a sequence of finite factors with $\dim M_n \to \infty$ and denote by M the ultraproduct II_1 factor $II_\omega M_n$, over a free ultrafilter ω on N. Let $Q \subset M$ be a von Neumann subalgebra satisfying one of the following:

(a) $Q = II_\omega Q_n$, for some von Neumann subalgebras $Q_n \subset M_n$ satisfying the condition $Q_n \not\prec M_n Q'_n \cap M_n$, ∀n (in [226]);

(b) $Q = B' \cap M$, for some separable amenable von Neumann subalgebra $B \subset M$.

Then given any separable subspace $X \subset M\theta\,(Q' \cap M)$, there exists adiffuse abelian von Neumann subalgebra $A \subset Q$ such that A is free independent to X, relative to $Q' \cap M, i.e\ E_{Q\cap M}(x_0 \prod_{i=1}^{n} a_i x_i) = 0$, for all $n \ge 1, x_\circ \in x \cup \{1\}, x_i \in X, a_i \in A\theta C_1 1 \le i \le n$.

Note that the particular case when $Q_n \subset M_n$ are II_1 factors with atomic relative commutant, for which one clearly has $Q_n \not\subset M_n Q'_n \cap M_n$, recovers (2.1 in [230]).

The conclusion in Theorem **(6.2.**1) above can alternatively be interpreted as follows: given any separable von Neumann subalgebra P of M that makes a commuting square with $Q' \cap M$ (in the sense of 1.2 in [110]) and we let $B_1 = P \cap (Q' \cap M)$, there exists a separable von Neumann subalgebra $B_0 \subset Q'$, such that $P \vee B_0 \simeq P *_{B_1} \left(B_1 \overline{\otimes} B_0\right)$ (amalgamated free product of finite von Neumann algebras over a common subalgebra, see [257], [251]). Since in the case (b) of Theorem **(6.2.**1) we have $Q' \cap M = B$ (see Theorem **(6.2.8)**) and all embeddings into an ultraproduct II_1 factor M of an amenable separable von Neumann algebra B are conjugate by unitaries in M, Theorem **(6.2.**1) shows in particular that if two separable finite von Neumann algebras N_1, N_2 containing copies of B are embeddable into M, then $N_1 *_B N_2$ is embeddable into M as well. Note that the case B atomic of this result already appears in [230], while the case B arbitrary but with $M = R^\omega$ was shown in [237]. Theorem **(6.2.**1) implies the following strengthening of these results:

Corollary (6.2.2) [233]: Let $M = II_\omega M_n$ be an ultraproduct II_1 factor as in Theorem **(6.2.**1). Let $N_i \subset M$ be separable finite $Q' \cap M$ von Neumann subalgebras with amenable von Neumann subalgebras $B_i \subset N_i, i = 1,2$,such that $(B_1, \tau_{\backslash B_1}) \simeq \left(B_2, \tau_{\backslash B_2}\right)$ Then there exists a unitary element $u \in M$ so that $uB_1u^* = B_2$ and so that, after identifying $B = B_1 \simeq B_2$ this way, we have $uN_1u^* \vee N_2 \simeq N_{1*B} N_2$.

To show Theorem **(6.2.**1), we first construct unitaries $u \in Q$ that are approximately n-independent with respect to given finite sets $X \perp Q' \cap M$. Taking larger and larger n, larger and larger finite sets X and better and better approximations, and combining with a diagonalization procedure, one can then get unitaries that are free independent to a given countable set, due to the ultraproduct framework.

The approximately independent unitary u is constructed by patching together incremental pieces of it, while controlling the trace of alternating words involving u and a given set X. This technique was initiated in [250], being then fully developed in [230], where it has been used to show a particular case of Theorem **(6.2.**1) (a). More recently, it has been used in [256] to establish existence of free independence in ultraproducts of maximal abelian $*$-subalgebras (abbreviated hereafter MASA) $A_n \subset M_n$ that are singular in the sense of [97] (i.e., any unitary element in M_n that normalizes A_n must lie in A_n), thus settling the Kadison-Singer problem for the corresponding ultrapower inclusion $\Pi_\omega A_n \subset \Pi_\omega M_n$.

If in turn the normalizers of the MASAs $A_n \subset M_n$ are large, then one can still detect certain independence properties inside A, by using the same type of techniques. Thus, it was shown in [256] that 3-independence always occurs in A, and we show here that given any countable group of unitaries Γ in M, that normalizes A and acts freely on it, there exists a diffuse subalgebra B_0 in A such that any word $\prod_{i=1}^{n} u_i b_i u_i^*$ with $b_i \in B_0 \theta\ \mathbb{C}$ and distinct $u_i \in \Gamma$, has trace 0. This actually amounts to B_0 being the base of a Bernoulli Γ -action, more precisely:

Theorem (6.2.3) [233]: Let $A_n \subset M_n$ be MASAs in finite factors, as before, and denote A = $A = \Pi_\omega A_n \subset \Pi_\omega M_n = M$. Assume $\Gamma \subset M$ is a countable group of unitaries nor- malizing A and acting freely on it, and let $H \subset \Gamma$ be an amenable subgroup. Given any separable abelian von Neumann subalgebra $B \subset A$, there exists a Γ-invariant subalgebra $A \subset A$ such that A, B are τ - independent and $\Gamma \curvearrowright A$ is isomorphic to the generalized Bernoulli action $\Gamma \curvearrowright L^\infty$ $([0,1])^{\Gamma/H}$.

Note that if the above ultraproduct inclusion $A \subset M$ comes from a sequence of finite dimensional diagonal inclusions $D_n \subset M_{n\times n}$,($\mathbb{C}$) or is of the form $D^\omega \subset R^\omega$, where $D \subset R$ is the unique (up to conjugacy by an automorphism, by [109]) Cartan subalgebra of the hyperfinite II_1 factor, then a countable group Γ can be embedded into the normalizer $N_M(A)$ of A in M, in a way that it acts freely on A , iff it is sofic (in [258], [249]). Thus, with the terminology in [238], where an action of a sofic group $\Gamma \curvearrowright X$ is called sofic if the inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ admits a commuting square embedding into $A \subset M$, with Γ embedding into $N_M(A)$, it follows from Theorem **(6.2.**3) that if $\Gamma \to X$ is sofic then any product action $\Gamma \curvearrowright X \times Y$ with $\Gamma \curvearrowright Y = [O,1]'$ a generalized Bernoulli action corresponding to the left action of Γ on a set $I = \oplus_i \Gamma / H_i$, for some countable family of amenable subgroups $H_i \subset \Gamma$, is sofic. This generalizes a result in [238].

We recall some basic facts needed, such as the local quantization lemma from [92], [229] and the criterion for (non-)conjugacy of subalgebras from [226]. We also show a general fact about centralizers (or commutants) of countable sets in ultraproduct II_1 factors. We show some bicentralizer results concerning amenable algebras and groups, in ultrapower framework, that we need in the proofs of Theorem **(6.2.**1) and respectively Theorem **(6.2.**3). We conjecture that, in fact, the bicentralizer property characterizes amenability.

We show the main technical result needed in the proof of Theorem**(6.2.**1) , by using incremental patching techniques. This result, stated as Lemma **(6.2.13)**, actually amounts to an "approximate version" of the free independence result in Theorem **(6.2.**1). We derive Theorem **(6.2.**1) (in fact a stregthening of it, stated as Theorem **(6.2.16)**, by using Lemma **(6.2.13)** and an appropriate diagonalization procedure.

We show Theorem **(6.2.**3) (stated as Theorem **(6.2.18)**. Also, we use the incremental patching technique to show (see Theorem **(6.2.20)** that if $A_n \subset M_n$ are Cartansubalgebras in finite factors, with dim $M_n \to \infty$, and Γ_i are countable subgroups of the normalizer N of $A = \Pi_\omega A_n$ in $M = \Pi_\omega M_n$, acting freely on A, with $H_i \subset \Gamma_i$ isomorphic amenable subgroups, then there exists $u \in N$ such that $uH_1u^* = H_2$ and such that the group generated by $u\Gamma_1 u^*$ and Γ_2 is the amalgamated free product $\Gamma_1 *_H \Gamma_2$, where H is the identification of H_1, H_2 via Ad(u). Taking M_n finite dimensional, this recovers a result from [239], [248], on the soficity of amalgamated free products of sofic groups over amenable subgroups and on the uniqueness of sofic embeddings of an amenable group.

All von Neumann algebras M considered are finite (in [118]) and come equipped with a fixed faithful normal trace state, generically denoted τ. We denote by $U(M)$ the group of unitary elements of M and by $P(M)$ the set of projections of M. Recall that a von Neumann algebra is a factor if its center is reduced to the scalars. Recall that there exists a unique trace state on a finite factor ([162]). A finite factor M is either finite dimensional (in which case $M \simeq M_{n\times n}$ ($\mathbb{C}$) for some $n \geq 1$ with its unique trace state τ given by the normalized trace $tr = Tr/n$) or infinite dimensional. In this latter case, it is called a II_1 factor, and is characterized by the fact that the range of the trace on the set of projections satisfies $\tau(p(M)) = [0,1]$.

More generally, a finite von Neumann algebra splits as a direct sum $M = M_1 \oplus M_2$ with M_1 of type I (i.e. $M_1 \simeq \oplus_{n\geq 1} M_{n\times n}(C) \otimes A_n$, where A_n are abelian von Neumann algebras, possibly equal to 0) and M_2 of type II_1 (which by definition means M_2 has no type I summand).

We denote by $\|x\|_2 = \tau \left(x^* x\right)^{1/2}$, $x \in M$, the L^2 Hilbert-norm given by the trace. We denote by L^2M the completion of M in this norm. We often view M in its standard representation, acting L^2M on by left multiplication.

We will also use the L^2 norm $\| \ \|_1$ on M, defined by $\|x\|_1$:= τ (|x|) = sup $\{|\tau(xy)| \mid y \in M, \|y\| \leq 1\}$. We denote by L^1M the completion of M in the norm $\| \ \|_1$. Note that by the Cauchy-Schwartz inequality we have $\|x\|_1 \leq \|x\|_2$, while by the inequality $x^*x \leq \|x\||x|$ we have $\|x\|_2^2 \leq \|x\|_1 \|x\|$.

If $M \subset B$ is a von Neumann subalgebra, then $E_B : M \to B$ denotes the (unique) τ - preserving conditional expectation of M onto B, which is contractible in both the operatorial norm $\| \ \|$ and the above L^P -norms, $p = 1,2$. If we view M in its standard representation on L^2M, then the expectation E_B is implemented by the orthogonal projection e_B of L^2M onto $L^2B \subset L^2M$ (viewed as the closure in the normof $B \subset M$), b$\| \ \|_2$ y $e_B x e_B = E_B(x) e_B, x \in M$ .Given avon Neumann subalgebra $B \subset M$ and aset $X \subset M$,we say that X is perependicular to B and write $X \perp B$ if $\tau(x^* b) = 0, \forall x \in X \ and \ b \in B.$

A finite von Neumann algebra (M, τ) is separable if it is separable withrespect to the Hilbert norm $\| \ \|_2$. Note that this condition is equivalent to thefactthat M is countably generated as a von Neumann algebra. More generally, $X \subset M$ is asubspace, then X is separable if it is seprable with respect to the norm$\| \ \|_2$.

The von Neumann algebra M is atomic if $1_M = \sum_i e_i$ with $e_i \in M$ a family of mutually orthogonal minimal projections $e_i \in M$ (or equivalently, atomic projections, i.e. with the property that $e_i M e_i = \mathbb{C} e_i$. M is diffuse if it has no minimal (non zero) projection. Any abelian von Neumann algebra A which is diffuse and separable is isomorphic to $L^\infty([0,1])$ (or to $L^\infty(\mathbb{T})$). Moreover, if A is endowed with a faithful normal state τ , then the isomorphism $A \simeq L^\infty([0,1])$ can be taken so that to carry τ onto the integral $\int \cdot\, d\mu \cdot$, where μ is the Lebesgue measure on [0, 1].

We will often consider maximal abelian ∗-subalgebras (MASA) A in a finite von Neumann algebra $M, i.e.$ M, i.e. abelian ∗-subalgebras $A \subset M$ with $A' \subset M = A$. In such a case, we denote $N_M(A) = \{u \in U(M) | uAu^* = A\}$, the normalize of A in M. Following [243], if the normalizer generates M as a von Neumann algebra, we call A a Cartan subalgebra in M. An isomorphism of Cartan inclusions $(A_0 \subset M_0; \tau) \simeq (A_1 \subset M_1; \tau)$ is a trace preserving isomorphism of M_0 onto M_1 carrying A_0 onto A_1.

If $A_0 \subset M_0$ is Cartan and $A_1 \subset M_1$ is an arbitrary MASA, then a Cartanembedding (or simply an embedding) of $A_0 \subset M_0$ into $A_1 \subset M_1$ is a trace preserving embedding of M_0 into M_1 that carries A_0 into A_1 such that $M_0 \cap A_1 = A_0$, with the commuting square condition $E_{A1} E_{M_0} = E_{A_0}$ satisfied (see Theorem (6.2.8)), and such that $N_{M_0}(A_0) \subset N_{M_1}(A_1)$.

For various other general facts about finite von Neumann algebras, see [162].

. Two von Neumann subalgebras $B_1, B_2 \subset M$ are in commuting square position if the expectations E_{B_1}, E_{B_2} commute (see Sec. 1.2 in [104]). Note that if this is the case then we in fact have $E_{B_1}, E_{B_2} = E_{B_2}, E_{B_1} = E_{B_1 \cap B_2}$. Also, for this to happen it is sufficient that $E_{B_1}(B_2) \subset B_1 \cap B_2$.

A typical example when the commuting square condition is satisfied is the following: let $Q \subset P \subset M$ be von Neumann algebras; then P and $Q' \cap M$ are in commuting square position (see 1.2.2 in [104]).

We notice here an observation showing that in the statement of Theorem**(6.2.**1) , we may equivalently take the space X to be a separable von Neumann algebra making a commuting square wit $Q' \cap M$, a fact that we will not use in the sequel but is good to keep in mind. See also (3.8 in [256]) for a similar statement.

Lemma (6.2.4)[233]: Let $N \subset M$ be a von Neumann subalgebra in the finite von Neumann algebra M. If $X \subset M$ is a separable subspace, then there exists a separable von Neumann subalgebra $P \subset M$ that contains X and makes a commuting square with N.

Proof: We let $P_0 \subset M$ be the (separable) von Neumann algebra generated by X and then construct recursively a increasing sequence of inclusions of separable von Neumann algebra $B_n \subset p_n, n \geq 1$, by letting B_n be the von Neumann algebra generated by $E_N\ (P_{n-1})$ and p_n be the von Neumann algebra generated by B_n and P_{n-1}.

If we now define $B = \overline{U_n B_n}^{\omega}$ and $P = \overline{U_n B_n}^{\omega}$, then both algebras are separable and $B \subset P \cap N$, by construction. Moreover, we have $E_N(P_n) \subset B_{n+1} \subset P$, implying that $E_N(P) \subset B \subset P \cap N$,i.e.N,P make a commuting square with $B = N \cap P$.

.An important example of a (separable) II_1 factor is the hyperfinite II_1 factor R of Murray and von Neumann ([118]), defined as the infinite tensor product $(R,\tau)=(R,\tau)=\overline{\otimes}_k(M_{2\times2}(C),tr)_k$. By [118], R is the unique approximately finite dimensional (AFD) separable II_1 factor (a separable finite von Neumann algebra algebra (M,τ) is AFD if there exists an increasing sequence of finite dimensional von Neumann subalgebras $M_n \subset M$ such that $U_n M_n$ is dense in M in the norm $\| \|_2$).

By Connes' results in [96], R is in fact the unique amenable separable II_1 factor. Recall in this respect that a finite von Neumann algebra (M,τ) is called amenable if there exists a state φ on $B(L^2M)$ that has M (when viewed in its standard representation on L^2M)in itscentralizer, $\varphi(xT)=\varphi(Tx), \forall x\in M, \forall T\in B(L^2M)$, and such that $\varphi \setminus M = \tau$. Note that the latter condition is redundant in case M is a factor, because $\varphi \setminus M$ is a trace and because of the uniqueness of the trace on factors. Connes Fundamental Theorem in [96] actually shows that amenability is equivalent to the AFD property, for any finite von Neumann algebra.

From all this, it follows that R can be represented in many different ways, for instance as the group measure space II_1 factor $L^\infty(X)\rtimes\Gamma$, associated with a free ergodic measure preserving action of a countable amenable group Γ ona probability space (X,u) ([118]). When viewed this way, R has $D=L^\infty(X)$ as a natural Cartan subalgebra. By [109], [246] the Cartan subalgebra of R is in fact unique, up to conjugacy by an automorphism of R. We may thus represent $D\subset R$ as the infinite tensor product $\overline{\otimes}_k(D_2)_K \subset \overline{\otimes} M_{2\times2}(\mathbb{C}))_k$,where D_2 is the diagonalsubalgebra in $M_{2\times2}(\mathbb{C})$.

More generally, by [97], if $A_0\subset R_0$ is a Cartan subalgebra in an amenable separable finite von Neumann algebra R_0 , then there exists an increasing sequence of finite dimensional Cartan inclusions ($A_{0,n}\subset R_{0,n})\subset(A_0\subset R_0)$ (with Cartan embeddings, as defined before) such that $\overline{U_n A_{0,n}}^{\omega}=A_0\subset R_0=\overline{U_n R_{0,n}}^{\omega}$.

We recall here a result from [92], [229], showing that if $Q\subset M$ are II_1 von Neumann algebras, then one can "simulate" the expectation onto the commutant $Q'\cap M$ by "squeezing" with appropriate projections in Q, a phenomenon called "local quantization" in [229]:

Theorem (6.2.5)[233]: (i) Let M be a finite von Neumann algebra and $Q\subset M$ a von Neumann subalgebra. Given any finite set $F\subset M\,\theta\, Q\vee$ ($Q'\cap M$) and any $\varepsilon>0$, there exists a projection $q\in Q$ such that $\|qxq\|_1<\varepsilon\tau(q), \forall X\in F$.

(ii) Let $Q\subset M$ be $Q'\cap M$ an inclusion of II_1 von Neumann algebras. Given any finite set $X\subset M$ and any $\varepsilon>0$, there exists a projection $q\in Q$ such that $\|qxq-E_{Q'\cap M}(x)q\|_1<\varepsilon\tau(q), \forall x\in X$. Moreover, q can be taken so that to have scalar central trace in Q.

Proof: Part (i) is already showd in [92] (see also Theorem 3.6 in [256]), while part (ii) is (Theorem A.1.4 in [229]).

Let $Q,P\subset M$ be von Neumann subalgebras of the finite von Neumann algebra M. Following [226], we say that a corner of Q can be embedded into P inside M and write $Q\prec_M P$ if the following condition holds true: there exist non-zero projections $p\in P, q\in Q$, a

unitalisomorphism $\psi : qQq \to pPp$ (not necessarily onto) and a partial isometr $v \in M$ such that $vv^* \in (qQq)' \cap qMq, v^*v \in \psi(qQq)', \cap pMp, xv = v\psi(x), \forall x \in qQq, and\ x \in qQq, xvv^* = 0 implies\ x = 0$.
We will actually consider cases when the above condition is not satisfied. We recall from (2.1 in [229]) a useful necessary and sufficient criterion for this to happen:
Theorem (6.2.6): Let M be a finite von Neumann algebra and $P, Q \subset M$ von Neumann subalgebras. For each $q \in P(Q), fix\, u_q \subset u(qQq)$ a subgroup generating qQq as a von Neumann algebra. Then $Q \nprec_M$ P if and only if the following condition holds true:
Given any $q \in p(Q)$ and any separable subspace $X \subset M$ there exists a sequence of unitary elements $u_n \in U_q$ such that $\lim_n \|E_P(xu_n y)\|_2 = 0, \forall x, y \in X$.

We fix once for all an (arbitrary) free ultrafilter ω on ℕ. If $M_n,\ n \geq 1$, is a sequence of finite von Neumann algebras then, we denote by $\Pi_\omega Mn$ their ω-ultraproduct, i.e., the finite von Neumann algebra obtained as the quotient of $\oplus_n M_n$ by its ideal $T_\omega = \{(x_n) \mid \lim_\omega \tau(x_n^* x_n) = 0\}$, endowed with the trace $\tau(y) = \lim_\omega \tau(y_n)$, where $(y_n)_n \in \oplus_n M_n$ is in the class y ∈ $y \in \oplus_n M_n / I_\omega$ ([259]).

Recall that if M_n are factors and dim $M_n \to \infty$, then $\Pi_\omega M_n$ is a II_1 factor ([259])and it is non-separable ([259]).

If $Q_n \subset M_n$ are von Neumann subalgebras, $n \geq 1$, then the ultraproduct $II_\omega Q_n$ identifies naturally to a von Neumann subalgebra in $\Pi_\omega Mn$ and its centralizer (or commutant) in $\Pi_\omega M_n$ is given by the formula $(\Pi_\omega Q_n)' \cap \Pi_\omega M_n = \Pi_\omega (Q_n' \cap M_n)$ (see e.g. [92]).

If M is a finite von Neumann algebra, then M^ω denotes its ω-ultrapower, i.e. the ultraproduct of infinitely many copies of M. Note that M naturally embeds into M^ω, as the von Neumann subalgebra of constant sequences, and that if M is a II_1 factor then M^ω is a (non-separable by [242]) II_1 factor.

Let $S = \{b_n\}_n$ be a countable subset in the ultrapower R^ω of the hyperfinite II_1 factor R and let $b_n = (b_{n,m})_m$ be representations of each of its elements with $b_{n,m} \in R = \overline{\otimes}_k (M_{2\times 2}(C))_k = \overline{\cup_r M_n}^{\omega}$, where M_n is the tensor product of the first n copies of $M_{2\times 2}(C)$. Thus, we may assume that for for each m, $\{b_{n,m}\}_{n \leq (m)} \subset M_{k_m}$, for a large enough k_m. Then we have $b_n \in \Pi_\omega M_{k_m} \subset R^\omega$, ∀n, viewed as a subalgebra of R^ω.But then the ultra product subalgebra $\Pi_\omega (M_{k_m}' \cap R) \simeq R^\omega$ commutes with the set $\{b_n\}_n$. This shows that the centralizer of any separable von Neumann subalgebra of R^ω is a type II_1 von Neumann algebra without separable direct summands.

However, for general ultra products $II_\omega M_n$ and ultra powers M^ω, we may have countable (or even finite) subsets S that have trivial centralizer: For instance, if M is a non-Gamma II_1 factor ([118]), such as the group II_1 factor $M = L(\Gamma)$ assoiated with an infinit conjugacy(ICC) countable group Γ with the property(T) of Kazhdan (for example, $\Gamma = PSL(n, Z), n \geq 3$). Then M is finitely generated and $M' \cap M^\omega = \mathbb{C}$.Similarly, by results in [234], it follows that if for some fixed $n \geq 3$ we take (π_m, H_m) to be any sequence of finite dimensional irreducible representations of $\Gamma = PSL(n, Z)$ so that $K_m = \dim H_m \to \infty$,then the von

Neumann subalgebra M generated by $\{(\pi_m(g))_m \mid g \in \Gamma\}$ in the ultraproduct II_1 factor $\Pi_\omega M_{K_m \times k_m}(\mathbb{C})$ isomorphie to the group factor $L(\Gamma)$ and has trivial relative commutant.

The following result shows that in fact the centralizer of a any separable von Neumann subalgebra P of an arbitrary ultraproduct II_1 factor $M := \Pi_\omega M_n$, coming from a sequence of finite factors M_n with dim $M_n \to \infty$, splits as the direct sum of an atomic von Neumann algebra and a diffuse von Neumann algebra with only non-separable direct summands.

Theorem (6.2.7)[233]: If P is a separable von Neumann subalgebra of M then $q' \cap M = B_0 \oplus B_1$, with B_0 atomic and B_1 diffuse and having no separable direct summand (even more: any MASA of B_1 has only non-separable direct summands).

Proof: Denote $Q = P' \cap M$ and let $z \in Z(Q)$ be the maximal central projection with the property that Q_Z is diffuse. We have to show that $Q_Z{}'$ is non-separable for any central projection $z' \in Z(Q)_Z$. By replacing $P \subset M$ by $P_Z \subset zMz$, we may clearly assume $z = 1$.

Assuming by contradiction that Q has separable direct summands, we may further reduce with the maximal central projection z_0 in Q with the property that Q_{Z_0} is separable to actually assume, by contradiction, that $P \subset M$ is separable with $Q = P' \cap M$ diffuse and separable.

Let $\{b_n\}_n \subset P$ be a countable subset of the unit ball of P, dense in the Hilbert norm$\| \ \|_2$ .Let $b_n = (b_{n,m})_m$ be representations of b_n with $b_{n,m} \in M_m$, $\|b_{n,m}\| \le \|b_n\| \forall n,m$. Let also $u \in Q$ be a Haar unitary generating a maximal abelian $*$-subalgebra A_0 $of\, Q$, and let $u = (u_m)_m$ be a representation of u with $u_m \in U(M_m), \forall m$.

The fact that u belongs to $Q = \{b_n\}'_n \cap M$ translates into the condition

$$\lim_{m\to\omega} \| [b_{k,m}, u_m] \|_2 = 0, \forall k \ge 1, \tag{6}$$

While the fact that u is a Haar unitary amounts to the condition

$$\lim_{m\to\omega} \tau(u_m^j) = 0, \forall j \ne 0,. \tag{7}$$

Let V_n denote the set of all $m \in N$ with the property that

$$\| [b_{k,m}, u_m] \|_2 < 2^{-n}, |\tau(u_m^j)| < 2^{-n}, \forall 1 \le k \le n, 1 \le |j| \le 2n. \tag{8}$$

If we identify $\ell^\infty \mathbb{N}$ with the algebra $C(\Omega)$ of continuous functions on its spectruma (via the GNS representation), and we view ω as a point in Ω, then by (6) and (7)it follows that V_n correspond to an open-closed neighborhoods of $\omega \in \Omega$. Let now $W_n, n \ge 0$, be defined recursively as follows: $W_0 = \mathbb{N}$ and $W_{n+1} = W_n \cap V_{n+1} \cap \{n \in N \mid n > \min W_n\}$. Note that, with the same identification as before, W_n correspond to a strictly decreasing sequence of neighborhoods of ω .

Noticing that the sets $\{W_n \setminus W_{n-1}\}_{n>1}$ form a partition of N, we define $v = (v_m)_m$ by letting $v_m = u_m^n$ for $m \in W_{n-1} \setminus W_n$. Since $v_m \in U(M_m)$, it follows that v is a unitary element in M . By the first relation in (15), if $m \in W_n \setminus W_{n-1}$ then

$$\| [b_{k,m}, u_m] \|_2 = \| [b_{k,m}, u_m^n] \|_2 \le \sum_{j=0}^{n-1} \| u_m^j [b_{k,m}, u_m] u_m^{n-j-1} \|_2 \le n 2^{-n}, \tag{9}$$

for all $1 \le k \le n$, while by the second relation in (8) we have

$$\left|\tau(v_m u_m^j)\right| 2^{-n} \tag{10}$$

for all $1 \le |j| \le n$.

But then (9) implies $v \in \{b_n\}' n \cap M = P' \cap M = Q$, while by (10) we have $\tau(vu^j) = 0$, for all $j \neq 0$, i.e. $v \in Q$ is perpendicular to the maximal abelian $*$-subalgebra $A_0 = \{u\}''$ of Q generated by $u \in Q$. Since by construction we have $uv = vu$, this shows that at the same time we have $v \in \{u\}' \cap Q = A_0$ and $v \perp A_0$,a contradiction. This also shows the stronger form of the statement.

Theorem (6.2.8) [233]:(i) Let M_n be a sequence of finite factors with dim $M_n \to \infty$ and denote $M = \Pi_\omega M_n$. If $B \subset M$ is a separable amenable von Neumann subalgebra, then $(B' \cap M)' \cap M = B$. Moreover, $B' \cap M$ is of type II_1 and has only non-separable direct summands.

(ii) If R denotes the hyper finite II_1 factor then $(R' \cap R^\omega)' \cap R^\omega = R$.

Proof: Part (ii) is just a particular case of part (i), so we only need to show (i). By Cannes' Theorem ([96]), since B is amenable and separable, it is approximately finite dimensional, so $B = \overline{U_n B_n}^\omega$, for some increasing sequence of finite dimensional von Neumann subalgebras $B_n \subset B$. Note that $B' \cap M = \cap_n (B_n' \cap M)$ and that for each n we have $(B_n' \cap M)' \cap M = B_n$(in fact, it is trivial to see that given any inclusion of von Neumann algebras $N \subset M$ with dim $N < \infty$ and M a factor, we have $(N' \cap M)' \cap M = N)$. We first need to show the following:

Fact. Let $P \subset M$ be an inclusion of finite von Neumann algebras. Let $x \in M \,\theta\, (P' \cap M)$ and $\varepsilon > 0$. There exists a unitary element $u \in P$ such that $\Re\tau(x^* u x u^*) \le \varepsilon \|x\|_2^2$.

To show this, let K_x denote the weak closure of the convex set co$\{uxu^* \mid u \in U(P)\}$ and note right away that $\|y\| \le \|x\|$ and $\|y\|_2 \le \|x\|_2, \forall y \in K_x$. Thus, K_x is a weakly closed bounded subspace in both M and $L^2 M$. In particular, there exists a unique element $y_0 \in K_x$ of minimal Hilbert-norm: $\|y_0\|_2 \min\{\|y\|_2 \mid y \in k_x\}$. Since K_x is $A\,\mathrm{d}\,(U(P))$-invariant (because it is the weak closure of the $A\,\mathrm{d}\,(U(P))$- invariant set co $\{uxu^* \mid u \in U(P)\}$ and since $\|uy_0u^*\|_2 = \|y_0\|_2$, by the uniqueness of y0 it follows that $uy_0u^* = y_0, \forall u \in U(P)$. Thus, $uy_0 = y_0 u, \forall u \in (p)$. By taking linear combinations of u, this implies $y_0 \in P' \cap M$. But by its construction, the entire K_x lies in $M\,\theta(P' \cap M)$. Thus, y_0 is both in $P' \cap M$ and perpendicular to it, implying that $y_0 = 0, i.e. 0 \in K_x$.

Assuming now that we have $\Re\tau(x^* u x u^*) \ge \varepsilon \|x\|_2^2$, for all $u \in U(P)$, by taking convex combinations over $u \in U(P)$ and then weak closure, it follows that $\Re\tau(x^* y) \ge \varepsilon \|x\|_2^2$, for all $y \in P$. In particular, $0 = \Re\tau(x^* y_0\ ) \ge \varepsilon \|x\|_2^2$, forcing $x = 0$. This ends the proof of the above Fact.

Denote for simplicity $Q = B' \cap M$ and note that $B \subset Q' \cap M$. Assume there exists $x \in Q' \cap M$ with $x \perp B$. In particular $x \perp B_n = (B_n' \cap M)' \cap M$. By applying the Fact to the inclusion $B_n' \cap M \subset M$ and the element x, it follows that there exists a unitary element $u_n \in B_n' \cap M$ such that $\Re\tau(x^* u_n x u_n^*) < 2^{-n}, \forall n$.

Let $\{e_n^k\}_k \subset B_n$ denote the (finite) pseudo group of all partial is ometries in B_n that can be obtained as a sum of elements from a given matrix unit of B_n, and which we take so that $\{e_i^n\}_i$ is a subset of $\{e_j^{n+1}\}_j, \forall n.$ Let $e_k^n = (e_{k,m}^n)_m$, with $e_{k,m}^n \in M_m$ chosen so that $\|e_{k,m}^n\| \le \|e_k^n\|$ and

$\{e^n_{i,m}\}_i \subset \{e^{n+1}_{j,m}\}_j$ for all n,m. Let also $u_n = (u_{n,m})_m$, with $u_{n,m} \in U(M_m)$. Then the above properties translate into

$$\lim_{m\to\omega}\left\|\left[u_{n,m}, e^n_{k,m}\right]\right\|_2 = 0, \lim_{m\to\omega}\Re\tau\left(x^*_m u_{n,m} x_m u^*_{n,m}\right) < 2^{-n}, \quad (11)$$

Foral K and all n, where $x = (x_m)_m$ with $x_m \in M_m$. Let V_n denote the set of all $m \in N$ with the property that

$$\left\|\left[u_{n,m}, e^n_{k,m}\right]\right\|_2 < 2^{-n}, \Re\tau\left(x^*_m u_{n,m} x_m u^*_{n,m}\right) < \|x\|_2^2/2, \forall k. \quad (12)$$

By (11), it follows that V_n corresponds to an open-closed neighborhood of ω in the spectrum Ω of $\ell^\infty\mathbb{N}$, under the identification $\ell^\infty\mathbb{N} = C(\Omega)$. Let now $W_n, n \geq 0$, be defined recursively as follows: $W_0 = \mathbb{N} and\ W_{n+1} = W_{n+1} \cap \{n \in \mathbb{N} | n > minW_n\}$. Note that, with the same identification as before, W_n correspond to a strictly decreasing sequence of neighborhoods of ω . Define $v = (v_m)_m$ by letting $v_m = u_{n,m}$ for $m \in W_{n-1} \setminus W_n$. Since $v_m \in U(M_m)$, it follows that v is a unitary element in M, while by the first relation in (12) and the fact that $\{e^n_{i,m}\}_i \subset \{e^{n+1}_{j,m}\}_i$ it follows that $v \in \bigcap_n B'_n \cap M = B' \cap M = Q$. By the second relation in (12), we also have $\Re\tau\left(x^* v x v^*\right) \leq \|x\|_2^2 \setminus 2$. But $x \in Q' \cap M$ by our assumption, thus $vxv^* = x$, giving $\tau(x^* v x v^*) = \|x\|_2^2$, a contradiction.

If $Q = Q_Z + Q\,(1-z)$ with z a non-zero central projection of Q and Q_Z separable, then by the bi-commutant property we have $z \in B$ and by Proposition (6.1.17) Q_Z is atomic. Thus, $B_Z = (Q_z)' \cap zMz$ would follow non-separable, a contradiction.

Assume now that $Q = Q_Z + Q(1-z)$ with $z \in p(Z(Q))$ such that Q_Z is type I. By the b_i-commutation relation, it follows again that $z \in B$ and that $B_z = (Q_z)' \cap zMz$ is non-separable (because the commutant of any abelian von Neumann subalgebra of M is non-separable, by 4.3 in [92], or 2.3 in [256]).

Theorem (6.2.9) [233]: Let $A_n \subset M_n$ be a sequence of MASAs in finite factors and denote $A = \Pi_\omega A_n \subset \Pi_\omega M_n = M, N = N_M(A)$.

(i) If $H \subset N$ is a countable amenable subgroup, then $(H' \cap A)' \cap M = A \vee H$.

(ii) Assume the MASAs $A_n \subset M_n$ are Cartan. Let $R_0 \subset M$ be a separable amenable von Neumann subalgebra such that $D_0 = R_0 \cap A$ is a Cartan subalgebra in R_0 with $\mathcal{N}_{R_0}(D_0) \subset \mathcal{N}(i.e.\ (D_0 \subset R_0) \subset (A \subset M)$ is a cartan embeddeding, in the sense of 1.3) .Then $(N_{R_0}(D_0)' \cap N)' \cap N = N_{R_0}(D_0)$.

Moreover, if $D_1 \subset R_1$ is another Cartan inclusion witch is cartan embedded in into $A \subset M$, then given any isomorphism $\rho : (D_0 \subset R_0; \tau) \to (D_1 \subset R_1; \tau)$, there exists $u \in N$ such that $Ad(u) = p\ on\ R_0$.

Proof: (i) Let first $\{e^n_j\}_j$ be an increasing sequence of finite partitions in $p(A)$ such that $\lim_n \left\|\sum_j e^n_j u e^n_j - E_A(u)\right\|_2 = 0, \forall u \in H$ (e.g., by [98], or 3.3 in [297]). If we denote by A_0 the von Neumann subalgebra of A generated by $U_{u\in H^U}\{e^n_j \setminus j,n\}u*$ and $R_0 = A_0 \vee H$, then H normalizes A_0, A_0 is a Cartan subalgebra of R_0 and $A \vee H = A \vee R_0$. In particular, $H' \cap A = R'_0 \cap A$. Moreover, since H is amenable, R_0 follows amenable so by ([109], [246]) there exists an increasing sequence of finite pseudo groups of partial isometries $G_n = \{e_j^n\}_j$, normalizing A_0 (and A as well), with source and targets either equal or mutually orthogonal, for each n ≥ 1, and such tha $\{e_j^n \setminus j, n\}$ generate R_0.

It is then trivial to see that $H' \cap A = \cap_n (G_n' \cap A)$ and $(G_n' \cap A)' \cap A = G_n \vee A, A_n$. Then the rest of the proof proceeds with a "diagonalization" argument, exactly as at the end of the proof of Theorem **(6.2.8)**.

(ii) The proof of this part is similar to the one of Theorem **(6.2.8)**(i) and of Theorem **(6.2.9)(i)**.

(i) It is well known (and trivial to show) that if M_n is a sequence of finite factors with dim $M_n \to \infty$ and (B,τ) is a finite separable AFD von Neumann algebra, then there exists a trace preserving embedding $\theta_0 : B \hookrightarrow M \coloneqq \Pi_\omega M_n$ and that given any other such trace preserving embedding $\theta_1 : B \hookrightarrow M$, there exists a unitary element $u \in M$ such that $\theta_1(b) = u\theta_0(b)u^*, \forall b \in B$. In particular, any two copies of (B,τ) in M are unitary conjugate. By Connes' theorem [96], this means that the same holds true for any finite, separable, amenable B.

Moreover, by a result of K. Jung in [244], the converse is also true: if a finite separable von Neumann algebra (B,τ) has a unique (up to unitary conjugacy) embedding into either an ultraproduct $\Pi_\omega M_{n\times n}(\mathbb{C})$ or in R_ω, then B is amenable (see [230]). In fact, by a result of N. Brown in [236], if $B \subset R_\omega$ is non-amenable, then there exist uncountably many non conjugate copies of $B \, in \, R_\omega$.

Since given any ultraproduct II_1 factors $M = \Pi_\omega M_n$, all embeddings $B \hookrightarrow M$ of a given separable amenable finite von Neumann algebra are unitary conjugate in M, it seems interesting to investigate the converse in this general setting: is it true that if $B \subset M$ is a separable non-amenable von Neumann algebra of an arbitrary utraproduct II_1 factor, then there exist "many" non-conjugate copies of $B \;\, in \; M$? (see [241].)

On the other hand, related to Theorem **(6.2.8)** above, we propose the following new characterization of amenability for separable finite von Neumann algebras:

Conjecture (6.2.10)[233]: Let P be a separable finite von Neumann subalgebra of an ultraproduct II_1 factor M (notably, of $M = R^\omega$, or of $M = \Pi_\omega M_{n\times n}$ $(\mathbb{C})$. If the bicentralizer condition $(P' \cap M)' \cap M = P$ is satisfied, then P is amenable. In particular, if M is a separable II_1 factor such that $(M' \cap M^\omega)' \cap M^\omega = M \; then M \simeq R$.

Note that for a separable von Neumann subalgebra of an ultraproduct II_1 factor,conjecture is equivalent to the following statement:

Conjecture (6.2.11)[233]: Let P be a separable von Neumann subalgebra of an ultraproduct II_1 factor M. If P is the centralizer of a von Neumann subalgebra $Q \subset M$, i.e., $P = Q' \cap M$, then P is necessarily amenable.

Indeed, one clearly has that Conjecture (6.2.11) implies Conjecture (6.2.10). Assume in turn that Conjecture (6.2.10) holds true. Let $Q \subset M$ be so that $P = Q' \cap M$ is separable and denote $\tilde{Q} = P' \cap M$. Then we still have $\tilde{Q} \cap M = P$, so P satisfies the bicentralizer condition and it is separable, thus P is amenable.

Note also that the bicentraliser condition $(M' \cap M^\omega)' \cap M^\omega = M$ for a separable II_1 factor M, implies that M must be McDuff ([101]), i.e., it splits off the hyperfinite II_1 factor (or else

$M'\cap M^{\omega}$ is abelian, implying that the bicentralizer is non-separable), but that it cannot be of the form $N\overline{\otimes}R$, with N non-Gamma ([118]). In fact, if M has a II_1 von Neumann subalgebra $N\subset M$ satisfying the spectral gap condition $N'\cap M^{\omega}=(N'\cap M)^{\omega}$ ([255]), then M cannot satisfy the bicentralizer condition $(M'\cap M^{\omega})'\cap M^{\omega}=M$. Indeed, this is because taking bicentralizer is an operation preserving inclusions of algebras, and thus the bicentralizer of M in M^{ω} contains the bicentralizer of N in M^{ω}, which is equal to $((N'\cap M)^{\omega})'\cap M^{\omega}=N^{\omega}$. But the latter is non-separable, so it cannot be contained in M, which is separable.
(i) Since by ([109]), any Cartan inclusion $A_0\subset M_0$ with M_0 separable amenable finite von Neumann algebra is a limit of an increasing sequence of finite dimensional Cartan inclusions, it follows that any isomorphism between two embeddings of $A_0\subset M_0$ into an ultraproduct inclusion $A\subset M$ is implemented by a unitary element in $N_M(A)$. Indeed, this is clear for finite dimensional $A_0\subset M_0$, and the general case follows by a diagonalisation procedure.

If in turn $A_0\subset M_0$ is a Cartan subalgebra with M_0 non-amenable, and $A_0\subset M_0$ is embeddable into an ultraproduct $A\subset M$ which is either of the form $\Pi_{\omega}D_n\subset\Pi_{\omega}M_{n\times n}$ (ℂ), or of the form $D^{\omega}\subset R^{\omega}$, then any two copies of $A_0\subset M_0$ into $A\subset M$ that are conjugate by a unitary in $N_M(A)$, will have the corresponding copies of M_0 unitary conjugate in M. The procedure of constructing "many" non-conjugate embeddings of a non-amenable $M_0\subset M$ in the proof of (8.1 of [236]), is easily seen to actually give embeddings of $A_0\subset M_0$ into $A\subset M$. Thus, (8.1 in [266]) also implies that there exist uncountably many non-conjugate embeddings of $A_0\subset M_0$ into $A\subset M$. Altogether, this gives an analogue for Cartan inclusions (equivalently, for countable equivalence relations [243]), of K. Jung's characterization of amenability in [244], by a "unique embedding" - type property.

Part (ii) of Theorem **(6.2.9)** above suggests that, for a separable Cartan inclusion $A_0\subset M_0$ embedded into an ultraproduct of Cartan inclusions $A\subset M$, the bicentralizer property of the inclusion of full groups $N_{M_0}(A_0)\subset N_M(A)$ characterizes the amenability of $A_0\subset M_0$.

(iii) G. Elek and G. Szabo showd in [239] the following "unique embedding" type characterization of the amenability property for a countable group H, analogue to the one for finite separable von Neumann algebras in [244]: if H is amenable then any two embeddings of H into the normalizer N of $A=\Pi_{\omega}D_n\subset\Pi_{\omega}M_{n\times n}$ (ℂ) $= M$, acting freely on A, are conjugate by a unitary in N (this easily implies the same thing for $A=D^{\omega}\subset R^{\omega}=M$; note that by Corollary **(6.2.19)** below, the same "unique embedding" result actually holds true for ANY ultraproduct inclusion $A\subset M$);and that if H is sofic and non-amenable, then there exist at least two embeddings of H into N acting freely on A, non-conjugate by unitaries in N. In fact, as we mentioned, by (8.1 in [236]) there even exist uncountably many non-conjugate such embeddings.

Part (i) of Theorem **(6.2.9)** suggests the following alternative "bicentralizer" characterization of amenability for countable groups:

Conjecture (6.2.12): Let H be a countable group embeddable into the normalize of an ultraproduct MASA $A\subset M$ (notably $D^{\omega}\subset R^{\omega}$ *or* $\Pi_{\omega}D_n\subset\Pi_{\omega}M_{n\times n}$ (ℂ)), such that H acts freely on A and such that it satisfies the bicentralizer condition $(H'\cap A)'\cap M=A\vee H$. Then H is amenable.

Lemma (6.2.13)[233]: Let $Q \subset M$ be an inclusion of II_1 von Neumann algebras and assum $Q \nprec Q' \cap M$. Let $f \in Q$ be a non-zero projection. For any $n \geq 1$ and any $\varepsilon > 0$, there exists a partial isometry $v.$ in fQf suth that $v\,v^* = v^*v, \tau(v\,v^*) > \tau(f)/and\left\|E_{Q'\cap M}(x)\right\|_1 \leq \varepsilon, \forall x \in \bigcup_{k=1}^{n} F_{v.}^{k}$

Proof: It is clearly sufficient to show the statement in case $F = F^*$ and$\|x\| \leq 1, \forall x \in F$ Let $\delta > 0$. Denote $\varepsilon_0 = \delta, \varepsilon_k = 2^{k+1}\varepsilon_{k-1}, k \geq 1$.Denot

$$W = \{v \in fQf \mid vv^* = v^*v \in P(Q), \left\|E_{Q'\cap M}(x)\right\|_1 \leq \varepsilon_k \tau(v * v), \forall 1 \leq k \leq n, \forall x \in F_v^k\}.$$

Endow W with the order $\leq$ in which $w_1 \leq w_2$ iff $w_1 \leq w_2 w_1^* w_1$. $(W, \leq)$ is then clearly inductively ordered. Let v be a maximal element in W. Assume $\tau(v^*v) \leq \tau(f)/4$ and denote $P = f - v^*v$. Note that this implies $\tau(vv^*)/\tau(p) \leq 1/3$.

If w is a partial isometry in pQp with $q = ww^* = w^*w$ and we let $u = v + w$, then for $x = x_0 \prod_{i=1}^{k} u_i x_i \in F_u^k$ we have

$$x = x_0 \Pi_{i=1}^{k} = v_i x_i + \sum \ell \sum i z_{0,i} \Pi_{j=1}^{\ell} w_{i\,j} z_{j\,i}\,, \tag{13}$$

where the sum is taken over all $\ell = 1, 2, ..., k$ and all $i = (i_1, ..., i_\ell)$, with $1 \leq i_1 < ... < i_\ell \leq k$, and where $w_{i\,j} = w$ (resp. $w_{i\,j} = w^*$) whenever $v_{i\,j} = v$ (resp. $v_{i\,j} = v^*$), $z_{0,i} = x_0 v_1 x_1 ... x_{i1-1} p, z_{ji} = p x_{ij} v_{ij+1} ... v_{i\,j+1-1} x_{ij+1-1} p$, for $1 \leq j < \ell$, and $z_{\ell,i} = p x_{i_{\ell+1}} ... v_k x_k$.

By applying $E_{Q'\cap M}$ to the above equation, then taking $\|\ \|_1$ and applying triangle inequality, we then get:

$$\left\|E_{Q'\cap M}(x)\right\|_1 \leq \left\|(x_0 \Pi_{i=1}^{k} v_i x_i)\right\|_1 + \sum_\ell \sum_i \left\|(z_{0,i} \Pi_{j=1}^{\ell} w_{ij} z_{j,i})\right\|_1 \tag{14}$$

Since $v \in W$, the first term on the right side in (14)is majorized by $\varepsilon_k \tau(vv^*)$, so we are left with estimating the terms $z = z_{0,i} \Pi_{j=1}^{\ell} w_{ij} z_{j,i}$ in the double summation on the right hand side, which all have $\ell \geq 1$ number of appearances of powers of ω

We first deal with the terms where $\ell \geq 2$.

Since for $y_1, y_2, y \in M$ with $\|y_1\| \leq 1, \|y_2\| \leq 1$ we have $\left\|E_{Q'\cap M}(y_1 y y_2)\right\|_1 \leq \|y_1 y y_2\|_1 \leq \|y\|_1$, it follows that for any $\ell \geq 2$ we have:

$$\left\|E_{Q'\cap M}(z)\right\|_1 = \left\|E_{Q'\cap M}(z_{0i} w_{i1} z_{1,i} w_{i2} z_{2i} ... w_{i\ell} z_{\ell i})\right\|_1 \leq \left\|w_{i1} z_{1,i} w_{i2}\right\|_1 = \left\|q z_{1,i} q\right\| = \left\|q z_{1,i} q\right\|_{1pMp} \tau(p), \tag{15}$$

where $\tau_{pMp} = \tau(p)^{-1} \tau M$ and $\|\ \|_{1,pMp}$ denotes the L^1 -norm on pMp associated with this trace.

By applying Theorem (6.**1.4)** to the inclusion $pQp \subset pMp$ (with its trace τpMp) and to the finite set $X \subset pMp$ of all elements of the form $Z_{1,i} - E_{(Q'\cap M)_p}(z_{1,i}) \in pMp\theta\,(Q' \cap M)_p$, for some i = (i1, . . . , iℓ), $\ell \geq 2$, we obtain that for any α > 0, there exists $q \in P(pQp)$ such that

$$\left\|q z_{1,i} q - E_{(Q'\cap M)\,p}(z_{1,i}) q\right\|_{1,pMp} < \alpha \tau_{pMp}(q)\ . \tag{16}$$

Thus, by combining (15) and (16) we get

$$\begin{aligned}
\left\|E_{Q'\cap M}(z)\right\|_1 &< \left\|q z_{1,i} q\right\|_{1,pMp} \tau(q) \\
&\leq \left\|E_{(Q'\cap M)\,p}(z_{1,i}) q\right\|_{1,pMp} + \alpha \tau_{pMp}(q))\tau(p) \\
&= \left\|E_{(Q'\cap M)\,p}(z_{1,i})\right\|_{1,pMp\tau_{pMp}} (q)\tau(p) + \alpha \tau(q)
\end{aligned}$$

$$=\left\|E_{(Q'\cap M)p}(z_{1,i})\right\|_{1,pMp}\tau(p)+\alpha\tau(q)\,. \tag{17}$$

We now take into account that by the definition of the norm $\| \ \|_1$, we have

$$\left\|E_{(Q'\cap M)p}(z_{1,i})\right\|_{1,pMp}=\sup\{\left|\tau(yz_{1,i}\ )\right|/\tau(p)\mid y\in(Q'\cap M)_P\|y\|\le 1\}$$

$$=\sup\{|\,\tau(y(1-vv^*)x_{i1}v_{i1}+...v_{i2-1}x_{i2-1}(1-vv^*))\,|\,/\,\tau(p)\mid y\in Q'\cap M,\|y\|\le 1\} \tag{18}$$

But since $y\in Q'\cap M$ commutes with $v,1-vv^*\in Q$ and τ is a trace, we actually have $\tau(y(1-vv^*)x_i...x_{i2-1}(1-vv^*))=\tau(yx_{i1}.....x_{i2-1})-\tau(yv^*x_{i1}....x_{i2-1}v)$, so the last term in (18) is further majorized by

$$\sup\{\left|\tau(yx_{i_1}v_{i_1+1}...v_{i_2-1}x_{i_2-1})\right|/\tau(p)\mid y\in Q'\cap M\|y\|\le 1\}$$

$$+\sup\{\left|\tau(yv^*x_{i_1}v_{i_1+1}...v_{i_2-1}x_{i_2-1}v)\right|/\tau(p)\mid y\in Q'\cap M\|y\|\le 1\}$$

$$=\ (\left\|E_{Q'\cap M}(x_{i_1}v_{i_1+1}...v_{i_2-1}x_{i_2-1})\right\|_1$$

$$+\left\|E_{Q'\cap M}(v^*x_{i_1}v_{i_1+1}...v_{i_2-1}x_{i_2-1}\right\|_1/\tau(p). \tag{19}$$

Note at this point that $x_{i_1}v_{i_1+1}...v_{i_2-1}x_{i_2-1}$ lies in $F_v^{i_2-i_1-1}$ and $v^*x_{i_1}v_{i_1+1}...v_{i_2-1}x_{i_2-1}v$ lies in $F_v^{i_2-i_1+1}$. Also, $i_2-i_1+1\le k$, with the only case when $i_2-i_1+1=k$ corresponding to the case $i_1=1,i_2=k,l=2$, i.e., to the (single) term $z=x_0w_1(px_1v_2x_2...v_{k-1}x_{k-1}p)w_kx_k$ of the double summation in(14) . Thus by combining (17) and (19) and using that $\tau(vv^*)/\tau(p)\le 1/3$ and choosing $\alpha\le\delta/3$ (which is less than $(\varepsilon_j-\varepsilon_{j-2})/3,\forall j$, for this particular z we get

$$\left\|E_{Q'\cap M}(z)\right\|_1\le\varepsilon_{k-2}(\tau(vv^*)/\tau(p))\tau(q)+\varepsilon_k(\tau(vv^*)/\tau(p))\tau(q)+\alpha\tau(q)\le(\varepsilon_{k-2}/3+\varepsilon_k/3+\alpha)\tau(q)\le(2_{\varepsilon k}/\tau(q) \tag{20}$$

While for any z with $i_2-i_1+1\le k-1,$ we get

$$\left\|E_{Q'\cap M}(z)\right\|_1\le\varepsilon_{k-3}(\tau(vv^*)/\tau(p))\tau(q)+\varepsilon_{k-1}(\tau(vv^*)/\tau(p))\tau(q)+\alpha\tau(q)\le(\varepsilon_{k-3}/3+\varepsilon_{k-1}/3+\alpha)\tau(q)\le(2_{\varepsilon k-1}/\tau(q) \tag{21}$$

Since $2^{k+1}\varepsilon_{k-1}=\varepsilon_k$ and since there are $\sum_{i=2}^{k}\binom{k}{i}=2^k-k-1$ elements in the double sum in (13) for which $\ell\ge 2$, of which exactly one has $i_2-i_1+1=k$ and the rest satisfy $i_2-i_1+1\le k-1,$ by summing up (21) and (22), we get

$$\sum\nolimits_{\ell\ge 2}\sum\nolimits_i\left\|z_{0,i}\Pi_{j=1}^{\ell}w_{ij}z_{j,i}\right\|_1$$

$$\le(2^k-k-2)(2_{\varepsilon k-1}/3)\tau(q)+(2_{\varepsilon k}/3)\tau(q)$$

$$\varepsilon_k\tau(q)-(2k+4)(\varepsilon_{k-1}/3)\tau(q)\ . \tag{22}$$

Finally, from the double sum on the right hand side of (14) we will now estimate the terms with $\ell=1$. These are terms which are obtained from $x_ov_1x_1v_2x_2...v_kx_k$ by replacing exactly one v_i by w_i, so they are of the form $z=z_{0,i}w_iz_{1,i}$ where

$$i=1,2,...k,\,z_{0,i}=x_0v_1x_1...v_{i-1}x_{i-1}p,\,z_{1,i}=px_iv_{i+1}...v_kx_k\,and w_i=w^s if\,v_i=v^s$$

Note that there are k such them.

One should notice at this point that in the above estimates we only used the fact that $w^*w=ww^*=q\in p(Q)$ and that it satisfies (16) for appropriate α. But we did not use so far the actual form of w. We will make the appropriate choice for w now, by making use of the condition $Q\not\prec Q'\cap M$. Indeed, by Theorem **(**6.2.1) (2.1 in [226]), this latter condition implies that for all $\beta > 0$ and all finite sets $Y_1 = Y_1^* \subset M \ominus Q' \cap M, Y_2 = Y_2^* \subset M$, there exists a unitary element $w \in qQq$ such that

$$\|E_{Q'\cap M}(y_1 \omega y_2)\|_1 < \beta \|E_{Q'\cap M}(y_2 \omega y_1)\|_1 < \beta \forall y_1 \in Y_1, y_2 \in Y_2. \tag{23}$$

Note that since Y_1, Y_2 are selfadjoint sets, by taking adjoints in (23), from these estimates we also get:

$$\|E_{Q'\cap M}(y_2 \omega^* y_1)\|_1 < \beta \|E_{Q'\cap M}(y_1 \omega y_2)\|_1 < \beta \forall y_1 \in Y_1, y_2 \in Y_2. \tag{24}$$

Denote by Z the set of elements of the form $x_o v_1 x_1 ... v_{i-1} x_{i-1} p, or\ px_i v_{i+1} ... v_k x_k$, for all possible choices arising from elements in $\bigcup_{k=1}^{n} F_v^k$. By applying (23), (24) to $\beta = \varepsilon_{k-1}\tau(q)/2k, n \geq 1\ and Y_2 = Z \cup Z^* \cup \{E_{Q'\cap M}(z) \mid z \in Z \cup Z^*\}, Y_1 = \{y_2 - E_{Q'\cap M}(y_2) \mid y_2 \in Y_2\}$, it follows that there exists $w \in U(qQq)$ such that

$$\|E_{Q'\cap M}((x_0 v_1 x_1 ... v_{j-1} x_{j-1} - E_{Q'\cap M}(x_0 v_1 x_1 ... v_{j-1} x_{j-1} p)) w_j x_j v_{j+1} ... v_k x_k)\|_1 \leq \varepsilon_{k-1}\tau(q)/2k, \tag{25}$$

$$\| (E_{Q'\cap M}((x_0 v_1 x_1 ... v_{j-1} x_{j-1}) w_j (x_j v_{j+1} ... v_k x_k) - E_{Q'\cap M}(px_j v_{j+1} ... v_k x_k)\|_1 \leq \varepsilon_{k-1}\tau(q)/2k \ . \tag{26}$$

Thus, for each element with $\ell = 1$ in the double summation $\sum_\ell \sum_{i\, z_{0,i}} \Pi_{j=1}^{\ell} w_{ij} z_{ji}$, in (13), i.e., of the form $x_o v_1 x_1 ... v_{i-1} x_{i-1} w_j x_j v_{j+1} ... v_k x_k$, we have the estimate:

$$\begin{aligned}
&\|E_{Q'\cap M}((x_0 v_1 x_1 ... v_{j-1} x_{j-1} w_j x_j v_{j+1} ... v_k x_k)\|_1 \\
&\leq 2\varepsilon_{k-1}\tau(q)/2k + \|E_{Q'\cap M}(x_0 v_1 x_1 ... v_{j-1} x_{j-1}) w_j E_{Q'\cap M}(x_j v_{j+1} ... v_k x_k)\|_1 \\
&\leq \varepsilon_{k-1}\tau(q)/k + \gamma
\end{aligned} \tag{27}$$

where γ is the minimum between

$$\|E_{Q'\cap M}((x_0 v_1 x_1 ... v_{j-1} x_{j-1}) q\|_1 = \tau(q) \|E_{Q'\cap M}(x_0 v_1 x_1 ... v_{j-1} x_{j-1})\|_1$$

and

$$\|qE_{Q'\cap M}((x_j v_{j+1} ... v_k x_k)\|_1 = \tau(q) \|E_{Q'\cap M}(x_j v_{j+1} ... v_k x_k)\|_1$$

Both elements $x_o v_1 x_1 ... v_{i-1} x_{i-1}, x_j v_{j+1} ... v_k x_k$. belong to some $F_v^{j,n}$ with $j \leq k-1$, and at least one of them with $j \neq 0$. Thus, by the properties of $v \in W$ and the assumption $\tau(vv^*) \leq \tau(f)/4$, we have $\gamma \leq \varepsilon_{k-1}\tau(vv^*)\tau(q) \leq \varepsilon_{k-1}\tau(q)/4$.

Hence, the last term in (27)is majorized by $\varepsilon_{k-1}\tau(q)/k + \varepsilon_{k-1}\tau(q)/4$ .Since there are k terms with $\ell \geq 1$, obtained by taking $j = 1,...,k$ by summing up over j in (27) and combining with (22), we deduce from(14)the following final estimate:

$$\begin{aligned}
\|E_{Q'\cap M}(x)\|_1 &\leq \|E_{Q'\cap M}(x_0 \Pi_{i=1}^{k} v_i x_i)\|_1 + \sum_\ell \sum_i \|E'_{Q'\cap M}(z_{0,i} \Pi_{j=1}^{\ell} w_{ij} z_{j,i})\|_1 \\
&\leq \varepsilon_k \tau(vv^*) + (\varepsilon_k - (2k+4)\varepsilon_{k-1}/3)\tau(q) + (k/4+1)\varepsilon_{k-1}\tau(q) \\
&\leq \varepsilon_k \tau(vv^*) + \varepsilon_k \tau(ww^*) = \varepsilon_k((v+w)(v+w)^*)
\end{aligned} \tag{28}$$

Since $u = v + w$ has also the property that $uu^* = u^* u$, it follows from (28) that $u \in W$. But this contradicts the maximality of $v \in W$.

We conclude that $\tau(v^* v) > \tau(f)/4$. If we now take $\delta \leq \varepsilon / 2^{n^2+1}$, then $\varepsilon_n = 2^{(n+1)(n+2)/2}\delta \leq \varepsilon$ and the statement follows.

We denote by $\mathcal{Q}_u$ the class of von Neumann subalgebras $Q \subset M$ which are of the form $Q = \Pi_\omega Q_n$, for some sub algebras $Q_n \subset M_n$, and have the property that condition $Q \not\prec_M Q' \cap M$. We denote by $\mathcal{Q}_b$ the class of von Neumann subalgebras $Q \subset M$ with the property that $Q' \cap M$ is separable and $(Q' \cap M)' \cap M = Q$.

The next result provides some properties and examples of algebras in these two classes.

Proposition (6.2.14) [233]: (i) If by $Q \in \mathcal{Q}_u$, then Q is of type II_1 .

(ii) If $Q_n \subset M_n$ are von Neumann subalgebras such that $Q_n \not\prec_{M_n} Q_n' \cap M_n \forall n$. , then $Q' = \Pi_\omega Q_n$ satisfies $Q_n \not\prec_M Q' \cap M$, and thus by $Q \in \mathcal{Q}_u$.

(iii) Assume m_n is an increasing sequence of positive integers of the form $m_n = d_n . k_n$, with $d_n . k_n \in$ ∈ ℕ. Let $M_n = M_{m_n \times m_n}$ (ℂ), with $p_n = M_{d_n \times d_n}$ (ℂ), $Q_n = M_{k_n \times k_n}$ (ℂ), viewed as subalgebras of M_n that commute and generate M_n. Then $Q = \Pi_\omega Q_n$, $p = \Pi_\omega Q_n$ satisfy the following properties: $Q' \cap M = P, P' \cap M = Q$; satisfies $Q \not\prec_M P$, (and thus $Q \in \mathcal{Q}_u$) if and only if $\lim_\omega d_n / k_n = 0$.

(iv) If $B \subset M$ is a separable amenable von Neumann subalgebra, then $Q := B' \cap M$ satisfies $Q' \cap M = B$. Thus $Q \in \mathcal{Q}_b$.

(v) If thus $Q \in \mathcal{Q}_b$ then Q is of type II_1, has no separable direct summand, and s $Q \not\prec_M Q' \cap M$ (the latter being separable).

Proof. (i) If an inclusion of finite von Neumann algebras $B \subset M$ is so that B is type I, then there exists a non-zero projection $e \in B$ such that eBe is abelian, implying that $eBe \subset (eBe)' \cap eMe$, thus $B <_M B' \cap M$. Since in our case we have $Q \not\prec_M Q' \cap M$,this shows that Q cannot have type I summands, thus Q is II_1 .

Part (ii) is an immediate consequence and of the fact that $Q' \cap M = \Pi_\omega (Q_n' \cap M_n)$ with $E_{Q' \cap M}(x) = (E_{Q_n' \cap M_n}(x_n))_n$, for $x = (x_n)_n \in M = \Pi_\omega M_n$.

Part (iii) is an easy exercise while part (iv) is a direct consequence of Theorem **(**6.2.1).

To show part (v), note that if $Q \in \mathcal{Q}_b$ then Q has no separable direct summand, by the same observation we have used in the proof of part (i).

Note that Conjecture (6.2.10) predicts that the class $\mathcal{Q}_b$, only consists of centralizers of separable amenable subalgebras of M , i.e.,of the examples Proposition **(6.2.14)**(iv) above.

Note that the case B atomic of Corollary **(6.2.17)**(ii) above has already been shown in [230], while the case B arbitrary but $M = R^\omega$ was shown in [237] (see also [240]).

A particular case when the assumptions in Corollary **(6.2.17)**(i) are satisfied, is when the subalgebra $P \subset M$ making a commuting square with $Q' \cap M$ is itself separable. But there are interesting non-separable examples as well, that may even allow obtaining free product with amalgamation over the entire $Q' \cap M$ (which is non-separable in case by $Q \in \mathcal{Q}_u$). For instance, if $U \subset U(M)$ is a countable group of unitaries normalizing $Q' \cap M$, then the von Neumann algebra P generated by U and $Q' \cap M$ satisfies all the conditions in Corollary **(6.2.17)**(i) with $B_1 = Q' \cap M$.

Note in this respect that one can alternatively take in the statement of Theorem **(6.2.16)** the separable space X to be of the form $X = P \ominus (P \cap Q' \cap M)$, for some separable von Neumann algebra P making a commuting square with $Q' \cap M$. Indeed, due to Lemma (6.2.8), the two versions follow equivalent.

Lemma (6.2.15) [233]: Let $Q \subset M$ be a von Neumann subalgebra lying in either the class $\mathcal{Q}_u$ or the class $\mathcal{Q}_b$. Let $f \in Q$ be a projection and $X \subset M \ominus (Q' \cap M)$ a countable set.

Then there exists a partial isometry v in fQf such that $vv^* = v^*v$, $\tau(vv^*) \geq \tau(f)/4\, and\, E_{Q' \cap M}(x) = 0 \forall x \in X_v^k, \forall k \geq 1$.

Proof: Let $X = \{x_k\} k \geq 1$ be an enumeration of X and denote $x_0 = 1$. By applying Lemma **(6.2.13)** to the inclusion of II_1 von Neumann algebras $Q \subset M$, the projection $f \in Q$, the positive constant $\varepsilon = 2^{-n}$ and the finite set $X_n = \{x_k \setminus k \leq n\}$, we get a partial isometry wn in fQf with the property that $w_n w_n^* = w_n^* w_n, \tau(w_n^* w_n) \geq \tau(f)/4$ and

$$\left\| E_{Q' \cap M}(x) \right\|_1 < 2^{-n}, \forall x \in \bigcup_{k \leq n} (X_n)_{\omega_n}^k. \tag{29}$$

Let $f = (f_m)_m$ be a representation of f with f_m projections. Let also $x_k = (x_{k,m})_m$ be a representation of x_k, with $x_{k,m} \in M_m, \|x_{k,m}\| \leq \|x_k\|, \forall k, m$, and $w_k = (w_{k,m})_m \in Q$ a representation of w_k with $w_{k,m}$ partial isometries satisfying $w_{k,m} w_{k,m}^* = w_{k,m}^* w_{k,m}, \leq f_m$.

Assume first that $Q = \Pi_\omega Q_n \in \mathcal{Q}_u$, ω in which case we may clearly also assume $f_m \in p(Q_m)$ and $w_{k,m} \in f_m Q_m f_m \forall k, m$. Noticing that if $y = (y_n) \in M$ then $E_{Q' \cap M}(y) = (E_{Q_n' \cap M_n}(y_n))_n$, it follows from (29) that

$$\lim_{m \to \omega} \left\| E_{Q_m' \cap M_m}(x_{j0,m} \Pi_{i=1}^k w_{n,m}^{\gamma i} x_{j i,m}) \right\|_1 < 2^{-n}, \tag{30}$$

For all $1 \leq k \leq n, x_{j0}, x_{jk} \in X_n \bigcup \{1\}, x_{ji} \in X_n, \gamma_i \in \{\pm 1\},$.

Let V_n be the set of all $m \in \mathbb{N}$ with the property that

$$\left\| E_{Q_m' \cap M_m}(x_{j0,m} \Pi_{i=1}^k w_{n,m}^{\gamma i} x_{j i,m}) \right\|_1 < 2^{-n}, \tag{31}$$

for all $1 \leq k \leq n, 1 \leq j_i \leq n\ for i \geq 1, 0 \leq j_0 \leq n, \gamma_i \in \{\pm 1\}$. By (30) it follows that V_n corresponds to an open-closed neighborhood of ω in Ω, under the identification $\ell^\infty \mathbb{N} = C(\Omega)$. Let now $W_n, n > 0$, be defined recursively as follows: $W_0 = \mathbb{N}\ and\ W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} | n\ min W_n\}$. Note that, with the same identification as before, W_n is a strictly decreasing sequence of neighborhoods of ω.

Define $v = (v_m)_m$ by letting $v_m = w_{n,m}\ for m \in W_{n-1} \setminus W_n$. It is then easy to check that v is a partial isometry in fQf satisfying all the required conditions.

Assume now that $Q \in \mathcal{Q}_b$,. Let $v\{y_\ell\}_\ell \subset Q' \cap M$ be a countable set dense in the unit ball of $Q' \cap M$ in the norm $\| \|_2$. Note that if $y_\ell = (y_{\ell,m})_m$ then $x = (x_n)_n \in M$ satisfies $x \in Q$ iff $\lim_{m \to \omega} \|[x_m, y_{\ell,m}]\|_2 = 0, \forall \ell$. Also, $x \perp Q' \cap M$ iff $\lim_{m \to \omega} \tau(x_m y_{\ell,m}) = 0, \forall \ell$. Moreover, if $\delta > 0$, then $\left\| E_{Q_m' \cap M_m}(x) \right\|_1 < \delta$ iff $\lim_{m \to \omega} |\tau(x_m y_{\ell,m})| \leq \delta, \forall \ell$.

With this in mind, from (29) it follows that the partial isometries $w_n = (w_{n,m})_m \in Q$ satisfy

$$\lim_{m \to \omega} |\tau((x_{j0,m} \Pi_{i=1}^k w_{n,m}^{\gamma_i} x_{ji,m}) y_{\ell,m})| \leq 2^{-n}, \tag{32}$$

for all $1 \le k \le n, x_{jo}, x_{jk} \in X_n \cup \{1\}, x_{ji} \in X_n$, $\gamma_i \in \{\pm 1\}, \forall i$ and for all $\ell \ge 1$. Also, the fact that w_n belongs to fQf is equivalent to

$$\lim_{m\to\omega} \|[w_{n,m}, y_{\ell,m}]\| = 0, \forall \ell; \lim_{m\to\omega} \| f_m w_{n,m} f_m - w_{n,m} \|_1 = 0 \tag{33}$$

Let V_n be the neighborhood of ω consisting of all $m \in \mathbb{N}$ with the property that

$$| \tau((x_{j0,m} \Pi_{i=1}^k w_{n,m}^{\gamma_i} x_{ji,m}) y_{\ell,m}) | < 2^{-n}, | \tag{34}$$

$\|[w_{n,m}, y_{\ell,m}]\|_2 < 2^{-n}; \| f_m w_{n,m} f_m - w_{n,m} \|_1 < 2^{-n};$

For all $\ell = 1,2,...,n$ as well as for all $1 \le k \le n, x_{jo}, \in X_n \cup \{1\}, x_{ji} \in X_n, \gamma_i \in \{\pm 1\}$ .Let further: $W_n \subset \mathbb{N}, n \ge 0$. Be defined recursively as follows: follows: $W_0 = \mathbb{N}\ and\ W_{n+1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} | n\ min W_n\}$. It follows that W_n are all neighborhoods of ω , that $W_n \subset \bigcap_{j \le b} V_j, W_{n+1} \subset W_n\ and W_{n+1} \ne W_n$.

We now define $v = (v_m)_m$, by letting $v_m = w_{n,m}\ if\ m \in W_{n-1} \setminus W_n$. By the way $w_{n,m}$ have been taken, v follows a partial is ometry with $vv^* = v^*v$, while by the second relation in (34) we have $v \in fQf$ and by the first relation in (34) we have $E_{Q' \cap M}(x) = 0 \forall x \in X_v^k, \forall k \ge 1$.

Theorem (6.2.16)[233]: Assume $Q \subset M$ is either in the class If by $\mathcal{Q}_u$, or $\mathcal{Q}_b$,. If $X \subset M \ominus (Q' \cap M)$ is a separable subspace, then there exists a diffuse von Neumann subalgebra $A \subset Q$ such that A is free independent to X , relative to $Q' \cap M$, more precisely $E_{Q' \cap M}(x_0 \prod_{i=1}^{n} a_i x_i) = 0\ for all\, n \ge 1\ and\ all\, x_i \in X, 1 \le i \le n-1, x_0, x_n \in X \cup \{1\}, a_i \in A \ominus \mathbb{C},\ 1 \le i \le k$.

Proof : We construct recursively a sequence of partial isometries $v_1, v_2, ... \in Q$ such that

(i) $v_{j+1} v_j^* v_j = v_j, v_j v_j^* = v_j^*\ and \tau(v_j v_j^*) \ge 1 - 1/2^j, \forall j \ge 1$.

(ii) $E_{Q' \cap M}(x) = 0, \forall x \in Y_v^k, \forall k \ge 1$.

Assume we have constructed v_j for $j = 1,...,m$. If v_m is a unitary element, then we let $v_j = v_m$ for all $j \ge m$. If v_m is not a unitary element, then let $f = 1 - v_m^* v_m \in Q$. Note that $E_{Q' \cap M}(x') = 0$, for all $x' \in X' \underline{\underline{def}} \cup_k Y_{v_m}^k$ x′ . Thus, if we apply Lemma **(6.2.15)** to $Q \subset M$, the projection $f \in Q$ and the countable set $\acute{X} \subset M \ominus (\acute{Q} \cap M)$, then we get a partial isometry $u \in fQf$, with $uu^* = u^*u$ satisfying $\tau(uu^*) \ge \tau(f)/2$ and $E_{Q' \cap M}(x) = 0$ for all $x \in \cup_k (X')_u^k$. But then $v_{m+1} = v_m + u$ will satisfy both (i) and (ii) for $j = m+1$.

It follows now from (i) that the sequence v_j converges in the norm $\| \, \|_2$ to a unitary element $v \in Q$, which due to (ii) will satisfy the condition $E_{Q' \cap M}, \forall x \in \cup_n X_v^n$. , Now, since Q is a II_1 von Neuman algebra Q contains a copy of hype finite II_1 factor, which in turn contains ahaar unitary $u_0 \in R$ .But then unitary $u = v u_0 v^*$ clearly satisfies the conditions required in part (a) of Theorem **(6.2.16)**.

Corollary (6.2.17)[233]: With the same assumptions and notations as in Theorem **(6.2.16)** above, we have:

(i) Let $P \subset M$ be a von Neumann subalgebra making a commuting square with $Q' \cap M$ and denote $B_1 = P \cap (Q' \cap M)$. Assume that $L^2 P$ is countably generated both as a left and as a right B_1 Hilbert module (equivalently, there exists a separable space $X \subset P$ such that $X \perp B_1$,

and $spXB_1$ and spB_1X are both $\| \|_2$-dense in $P \ominus$ B1). Then there exists a diffuse von Neumann subalgebra $B_0 \subset Q$ such that $P \vee B_0 \simeq P *_{B_1} (B_1 \overline{\otimes} B_0)$.
(ii) Let $N_i \subset M$ be separable von Neumann algebras, with amenable subalgebras $B_i, i = 1,2$, such that $(B_1, \tau) \simeq (B_2, \tau)$. Then there exists a unitary element $u \in M$ such that $uB_1u = B_2$ and such that, after identifying $B = B_1 \simeq B_2$ via $Ad(u)$, we have $N_1 \vee uN_2u^* \simeq N_{1*B}N_2$.
Proof: (i) Let $X_0 \subset P \ominus B_1$ be a separable subspace such that spX_0B_1 and spB_1X_0 are $\| \|_2$-dense in $P \ominus B_1$. By Theorem **(6.2.16)**, there exists a diffuse von Neumann subalgebra $B_0 \subset Q$ such that B_0 is free independent to X_0 relative to $Q' \cap M$. It is sufficient to show that $E_{Q' \cap M}(x_0 \Pi_i y_i x_i) = 0, for\, any\, x_0 \in X_0B_1 \cup \{1\}, x_i \in X_0B_1, y_i \in B_0 \ominus \mathbb{C}, 1 \leq i \leq n.$ But any element in X_0B_1 can be approximated arbitrarily well by a linear combination of elements in B_1X_0. The "coefficient" in B_1 of each one of these elements commutes with $y_i - 1$, so we can "move it to the left", being "swollen" by the $x_{i_1} \in X_0B_1$.Thus, in the end, it follows that it is sufficient to have $E_{Q' \cap M}(x_{0,0} \Pi_i y_i x_{0,i}) = 0, for\ x_{0,0} \in X_0 \cup \{1\}, x_{0,i} \in X_0, y_i \in B_0 \ominus \mathbb{C}_1$, which is indeed the case because B_1 is free independent to X_0 relative to $Q' \cap M$.

(ii) By the first part of, after possibly conjugating with a unitary $u_0 \in M$, we may assume the subalgebras B_1, B_2 coincide. Denote B this common algebra and let $Q = B' \cap M$, which by Theorem **(6.2.8)** satisfies $Q' \cap M = B$ and by Proposition **(6.2.14)**(iv) it belongs to $\mathcal{Q}_b$. Now apply Theorem **(6.2.16)** to Q and to the separable space $X = N_1 \ominus B + N_2 \ominus B$, to conclude that there exists a unitary element $u_0 \in Q$ such that uN_2u^* and N_1 generate the free amalgamated product $\simeq N_{1*B}N_2$.

The crucial step in proving Theorem **(6.2.16**) is Lemma **(6.2.13)**. The technique used in its proof consists of building unitaries u that are approximately n-independent with respect to certain finite sets, by "patching" together infinitesimal pieces of u. This technique was first considered in (2.1 of [250]), to show that given any countable set X in a finite von Neumann algebra M and any diffuse abelian von Neumann subalgebra $A \subset M$, there exists a Haar unitary $u \in A^{\omega}$ such that any word that alternates letters from X and $\{u^n \mid n \geq 1\}$, has 0-trace. This result was a key tool in proving that any derivation of a II_1 factor into the ideal of compact operators is inner, in [250].

The technique was substantially refined in [230], to show a particular case of the case $Q \in \mathcal{Q}_u$ of Theorem **(6.2.16)**, in which that $Q = \Pi_{\omega} Q_n \in \mathcal{Q}_u$ is so that $Q_n \subset M_n$ are II_1 sub factors with atomic relative commutant $Q_n' \cap M_n$ (which thus clearly satisfy $Q \nprec_{M_n} Q_n' \cap M_n$). The result in [230] had several applications over the years: Thus, it played an important role in developing reconstruction methods in Jones theory of subfactors in ([251], [252], [254]) and it led, in combination with ([257]), to the definition of amalgamated free product of inclusions of finite von Neumann algebras in [251]. It was also used to show key technical results in [223], [218] and to show that the free product of standard invariants of subfactors defined in ([235]) can be realized in the hyper finite II_1 factor R (see A.3 in [223] and [218]).

The same incremental patching method was used in [256] to show that if $A_n \subset M_n$ is a sequence of MASAs in II_1 factors, then the abelian von Neumann algebra

$A=\Pi_\omega A_n \subset \Pi_\omega M_n = M$ contains dffuse subalgebras B_0 that are τ -independent o any given separable subalgebra $B \subset A$ and 3-independent to any given countable set $X \subset M \ominus A$, i.e. any alternating word with at most 3 letters from X and 3 letters from $B_0 \ominus \mathbb{C}_1$ has trace 0 (see 0.2 in [297]). Moreover, if A_n are all singular (in the sense of [97], i.e. any unitary normalizing A_n is contained in A_n), then B_0 can be chosen to be free independent to X, relative to A, a fact that allowed settling the Kadison-Singer problem for ultraproducts of singular MASAs $A \subset M$ (see 0.1 in [256]).

A concrete example of a diffuse subalgebra B_0 in an ultraproduct MASA A satisfying the 3-independence property is the following: Let $\Gamma \curvearrowright X$ be an ergodic (but not necessarily free) measure preserving action of a discrete group Let Γ on a probability space (X,μ) and Let $\Gamma \curvearrowright Y = [0,1]^\Gamma$ be the Bernoulli Γ-action with diffuse base. Let $A = L^\infty(X) \otimes L^\infty(Y)$ with $\Gamma \curvearrowright A$ the product action. Let $\mathrm{M} = \mathrm{A} \rtimes \Gamma$ and $A = A^\omega \subset M^\omega = M$ If we take $B = L^\infty(X)$ and let $B_0 = 1 \otimes L^\infty([0,1]) \otimes 1 \subset L^\infty(Y)$ be the base of the Bernoulli action, viewed as a tensor component of the infinite tensor product $\Gamma^\infty(Y) = \otimes_{g\in\Gamma} (\Gamma^\infty([0,1]))_g$,then it is easy to see that B_0 is τ -independent to B and 3-independent with respect to $X = \{u_g \backslash g \in \Gamma\}$.

This construction can actually be recovered "asymptotically" inside any group measure space von Neumann algebra. Indeed, using the incremental patching technique, we will now show that (generalized) Bernoulli Γ-actions can be retrieved inside any free action of Γ on an ultrapower of measure spaces. More generally we have:

Theorem (6.2.18) [233]: Let $A_n \subset M_n$ be a sequence of MASAs in finite factors, with dim $M_n \to \infty$, and denote $A = \Pi_\omega A_n \subset \Pi_\omega M_n = M$. Assume $\Gamma \subset \mathcal{N}_M(\mathrm{A})$ is a countable group of unitaries acting freely on A and let $H \subset \Gamma$ be an amenable subgroup. Given any separable abelian von Neumann subalgebra $B \subset A$, there exists a separable diffuse abelian subalgebra $A \subset A$ such that: A, B are τ -independent, of Γ normalizes A, and the action of of Γ on A is isomorphic to the generalized Bernoulli action $\Gamma \curvearrowright L^\infty([0,1])^{\Gamma/\mathrm{H}})$.

Proof: Let $\{u_g \backslash g \in \Gamma\}$ be the unitaries in of Γ . Denote by of $g_0 = 1, g_1, g_2, \ldots \in \Gamma$ a set of representants of Γ/H. It is clearly sufficient to construct a Haar unitary w in A such that w commutes with $u_h \forall h \in H$, and such that B and $u_{gi}\{w^n \backslash n \in \mathbb{Z}\}u_{gi}^*, i = 0,1,2,\ldots$, are all multi-independent, in the sense that for any K, any non-zero integers n_j, distinct non-negative integers m_j, and any $b \epsilon B$, we have $\tau\left(b\Pi_{j=0}^k u_{gmj} w^{nj} u_{gmj}^*\right) = 0$.

Thus, we let A_0 be the subalgebra of all elements in A that are fixed by H and let $\{b_n\}_n$ be a $\|\|_2$-dense subset of the unit ball of B. If v is a partial isometry in A_0, then we denote by $F_{v,n}$ the set of all elements of the form $b\Pi_{j=0}^k u_{gmj} v^{nj} u_{gmj}^*$,where $1 \le i \le n, 1 \le k \le n, m_j$ are distinct integers beween 0 and n, and $1 \le |n_j| \le n$. We first show the following:
Fact. Given any $n \ge 1$ and any $\delta > 0$, there exists a Haar unitary $v \in A_0$ such that $|\tau(x)| \le \delta, \forall x \in F_{v,n}$.

To show this, let $w \coloneqq \{v \in A_0 \mid |\tau(x)| \le \delta\tau(v^*v), \forall x \in F_{v,n}, \tau(v^m) = 0, \forall m \ne 0\}$. Endow w with the order $\le$ in which $w_1 \le w_2$ iff $w_1 = w_2 w_1^* w_1$. $(w, \le)$ is then clearly inductively ordered. Let v be a maximal element in w . Assume $\tau(v^*v) < 1$ and denote $p =$

$1 - v^*v$. If $w \in A_0 p$is a partial isometry satifying $\tau(w^m) = 0, \forall m \neq 0$, and we denote$u = v + w$, then we have:

$b_i \Pi_{j=0}^{k} u_{gmj} u^{nj} u_{gmj}^* = b_i \Pi_{j=0}^{k} u_{gmj} v^{nj} u_{gmj}^* + \sum b_i \Pi_{j=0}^{k} u_{g_{m_j}} z_j^{nj} u_{g_{m_j}}^*$, (35) where $z_j \in \{v, w\}$ and the sum is taken over all possible choicesfor $z_j = v$ or $z_j = w$,with at least one occurrence of $z_j = w$ (thus, there are $2^{k+1} - 1$ many terms in the summation). We thus get the estimate

$$\left| \tau(b_i \Pi_{j=0}^{k} u_{gmj} u^{nj} u_{gmj}^*)\right|$$
$$\leq \left| \tau(b_i \Pi_{j=0}^{k} u_{gmj} v^{nj} u_{gmj}^*)\right| + \sum \left| \tau(b_i \Pi_{j=0}^{k} u_{gmj} z_j^{nj} u_{gmj}^*)\right|$$
$$\leq \delta\tau(vv^*) + \sum{}' \left| \tau\left(b_i \Pi_{j=0}^{k} u_{gmj} z_j^{nj} u_{gmj}^*\right)\right| + \sum{}'' \left| \tau\left(b_i \Pi_{j=0}^{k} u_{gmj} z_j^{nj} u_{gmj}^*\right)\right|. \quad (36)$$

where the summation $\sum{}'$contains the terms with just one occurrence of $z_j = w$ and $\sum{}''$ is the summation of the terms that have at least 2 occurrences of $z_j = w$.

Since A is abelian, the terms $u_{g_{m_j}} z_j^{n_j} u_{g_{m_j}}^*$ in a product can be permuted arbitrarily.

Thus, in each summand of $\sum{}''$ we can bring two of the occurrences of w so that to be adjacent, i.e., of the form $y_1 u_{g_{m_j}} w^{n_j} u_{g_{m_j}}^* u_{g_{m_i}} w^{n_i} u_{g_{m_i}}^* y_2$ .since $g_{m_i} \neq g_{m_j}$ for all $i \neq j$, by applying part (i) of Theorem **(**1.7) to $Q = A_O p$ and the finite

Set $F = \left\{u_{g_{m_j}}^* u_{g_{m_i}} \middle| i \neq j\right\} \perp A \vee H = A_0' \cap M$ it follows that for any α > 0, there exists a non-zero $q \in \rho(A_0 p)$ such that

$$\left\| q u_{g_{m_j}}^* u_{g_{m_i}} q \right\|_1 < \alpha\tau(q), \forall 0 \leq m_i \leq m_j \leq n . \quad (37)$$

Since there are $2^{k+1} - (k+1) - 1$ terms in the summation $\sum{}''$, this shows that $\sum{}'' < (2^{k+1} - (k+1) - 1) \propto \tau(q)$, for any choice of w that has support q q satisfying condition (37). Thus, if we choose $\alpha \leq 2^{-n-2}\delta$, then by (37) we get $\sum{}'' \leq \delta\tau(q)/2$.

So we are left with estimating the terms in the summation $\sum{}'$, which have just one occurrence of $w^j, j \neq 0$,i.e are of the form$\left|\tau(y_1 w^j y_2)\right| = \left|\tau(w^j E_A(q y_2 y_1 q))\right|$, for some $y_1, y_2 \in M, 1 \leq |j| \leq n$. There are $k + 1$many such terms for each $k = 1, \ldots, n$. Let's denote by Y_0 the set of all y_1, y_2 which appear this way, and note that this is a finite set in qMq. Thus $Y = E_A(qY_0.Y_0 q)$is finite as well.

It is sufficient now to find a Haar unitary $w \in A_0 q$ such that$\left|\tau(w^j y)\right| \leq \delta\tau(q)/2(n+1), \forall y \in Y, 1 \leq |j| \leq n$, because thenthe sum of the $k + 1$terms in $\sum{}'$will be majorized by $\delta\tau(q)/2$, altogether showing that for all $x \in F_{n,n}$, we have $|\tau(x)| \leq \delta\tau(uu^*)$. Since $A_0 q$ is diffuse, it contains a separable diffuse von Neumann subalgebra A_0, which is isomorphic to $L^\infty(\mathbb{T})$ with the Lebesgue measure corresponding to $\tau(q)^{-1}\tau_{A_0}$. Let then $w_0 \in A_0$be a Haar unitary generating A_0 . Since$\{w_0^m\}_m$ tends to 0 in the weak operator topology and $Y \subset q$is a finite set, there exists $n_0 \geq n$ such that $|\tau(w_0^m y)| \leq \delta\tau(q)/2(n+1)$, , for all $y \in Y$ and $|m| \geq n_0$. But then $w = w_0^{n_0}$ is still a Haar unitary and it satisfies all the required conditions.

This ends the proof of the Fact.

By using this Fact, it follows that for each n there exists a unitary element $v_n \in A_0$ such that

$$|\tau(x)| < 2^{-n}, \forall x \in F_{v_{n,},n}. \quad (38)$$

For each $g \in \Gamma$, let $u_g = (u_{g,m})_m$ be a representation of u_g with $u_{g,m} \in \mathcal{N}_{M_n}(A_n)$ .Let also $b_i = (b_{i,m})_m$an $v_n = (v_{n,m})_m \in A_0$, with $b_{i,m}, v_{n,m} \in A_m, \forall m$. Then (38) becomes

$$\lim_{m\to\omega} \left| \tau(b_{i,m} \Pi_{j=0}^{k} u_{g_{j,m}} v_{n,m} u^*_{g_{j},m}) \right| < 2^{-n} \quad (39)$$

for all $1 \le i, k \le n, 0 \le j_0 < j_1 \dots < j_k \le n$. Also, the fact that $v_n$lies in A_0 translates into

$$\lim_{m\to\omega} \left\| [u_{h,m}, v_{n,m}] \right\|_1 = 0, \forall h \in\leftarrow H, n \ge 1 \quad (40)$$

Let then v_n be the set of all $m \in N$ satisfying the following properties:

$$\left| \tau(b_{i,m} \Pi_{j=0}^{k} u_{g_{m_j,m}} v_{n,m} u^*_{g_{m_j},m}) \right| < 2^{-n} \quad (41)$$

$$\left\| [u_{h,m}, v_{n,m}] \right\| < 2^{-n}$$

For all $1 \le i, k \le n, 0 \le j_0 < j_1 \dots < j_k \le n$,where$\{h_i\}_i = H$ is an enumeration of H. Note that by (39) and (40), V_n corresponds to an open-closed neighborhood of ω in Ω , under the identification $\ell^\infty(\mathbb{N}) = C(\Omega)$. Define now recursively $W_0 = \mathbb{N}$ and $W_{n+\_1} = W_n \cap V_{n+1} \cap \{n \in \mathbb{N} | n > min\, W_n\}$ .Then W_n is a strictly decreasing sequence of neighborhoods of ω (under the same identification as before) with $W_n \subset \cap_{j\le n} V_j$.

We now define $w = (w_m)_m$, by letting $w_m = v_{n,m}$ if $m \in W_{n-1} \backslash W_n$. By the way $v_{n,m}$ have been taken, w follows unitary element in A, while by the second relation in (41) we have $w \in A^H = A_0$. Also, by the first relation in (41) it follows that B and $u_{g_i}\{w^n \backslash n \in \mathbb{Z}\}u^*_{g_i}, i = 0,1,2, \dots$, are all multi-independent. Thus, if we denote by $A \subset A$ the von Neumann algebra generated by $u_{g_i}\{w^n \backslash n \in \mathbb{Z}\}u^*_{g_i}, i \ge 0$, then A and B are τ -independent and $\Gamma \curvearrowright A$ is isomorphic to the generalized Bernoulli action $\Gamma \curvearrowright L^\infty([0,1]^{\Gamma\backslash H}$, as desired.

Corollary (6.2.19)[233]: As in Theorem (6.2.18), let $A_n \subset M_n$be a sequence of MASAs in finite factors, with $dimM_n \to \infty$, and denote $A = \Pi_\omega A_n \subset \Pi_\omega M_n = M$. Let $G \curvearrowright X$ be a measure preserving (but not necessarily free) action of a countable amenable group G on a probability space (X, μ) . Let $\rho_i: L^\infty(X) \rtimes G \hookrightarrow M$ be trace preserving embeddings taking $L^\infty(X)$ into A, with commuting squares, and G in the normalize $\mathcal{N}$ of A in M , such that $\rho_i(G)$ acts freely on $A, i = 1,2$. Then there exists $u \in \mathcal{N}$ such that $u\rho_1(x)u^* = \rho_2\,(x) \forall x \in L^\infty(X) \rtimes H$. In particular, any two embeddings of G into $\mathcal{N}$ acting freely on A , are conjugate by a unitary in $\mathcal{N}$.

Proof: By Theorem **(6.2.18)** applied to $\Gamma = G$ and $H = \{1\}$, each one of the embeddings ρ_i can be extended to embeddings, still denoted by ρ_i , of $A = L^\infty(X \times [0,1]^G) \subset L^\infty(X \times [0,1]^G) \rtimes G = M$ into $A \subset M$, satisfying the same properties, where$G \curvearrowright X \times [0,1]^G$is the product action. This action is free, so the corresponding inclusion is Cartan, with M AFD. Thus, by observation, the specific isomorphism $\rho_2 \circ \rho_1^{-1}: \rho_1(M) \simeq \rho_2(M)$is implemented by a uitary in $\mathcal{N}$.

Finally, let us mention that a slight adaption of the proof of Theorem (6.2.16) allows showing that given any two countable groups Γ_1, Γ_2 normalizing D^ω in R^ω (where as before $D \subset R$ is the Cartan subalgebra of the hyperfinite II_1 factor), there exists a unitary element $u \in \mathcal{N}_{R^\omega}(D^\omega)$that conjugates Γ_1 in free position with Γ_2. Moreover, if $H \subset \Gamma_1 \cap \Gamma_2$ is a common amenable group, then u can be taken so that to commute with H and so that the group Γ generated by $u\Gamma_1 u^*$ and Γ_2 satisfies $\Gamma \simeq \Gamma_{1*H}\Gamma_2$, with Γ acting freely if Γ_1, Γ_2 act freely. This recovers a result from [248], [239]. We'll actually state and show only the case Γ_i act freely of such a statement, for clarity:

Theorem (6.2.20) [233]: Let $A_n \subset M_n$ be a sequence of Cartan MASAs in finite factors, with $dimM_n \to \infty$, and denote $A = \Pi_\omega A_n \subset \Pi_\omega M_n = M$, as before. Assume $\Gamma_i \subset \mathcal{N}_M(A)$ are countable groups of unitaries acting freely on A, with amenable subgroups $\mathrm{H_i} \subset \Gamma_i, i = 1,2$, such that $\mathrm{H}_1 \simeq \mathrm{H}_2$. Then there exists a unitary element $u \in \mathcal{N}_M(A)$ such that $\mathrm{uH}_1 u^* = H_2$ and such that the group generated by $\mathrm{u}\Gamma_1 u^*$ and Γ_2 is isomorphic to $\Gamma_{1*H}\Gamma_2$ and acts freely on A, where H is the identification $\mathrm{H}_1 \simeq \mathrm{H}_2$ under $Ad(u)$.

Proof: By Corollary **(6.2.19)** above, there exists a unitary element $u_0 \in \mathcal{N}_M(A)$ such that $\mathrm{u_0 H}_1 u_0^* = H_2$. We may thus assume $\mathrm{H}_1 = H_2$, a common subgroup we will denote by H.
Denote $A_0 = H' \cap A$ .Let also $\mathcal{N}_0 = H' \cap A$ and note that $\mathcal{N}_0$ normalize $s\, A_0$. since by Theorem (6.2.9) we have $A_0' \cap M = A,$ it follows that A_0 is a MASA in $M_0 = A_0 \vee \mathcal{N}_0$ and that $\mathcal{N}_0$ is the normalize$s\, A_0$ in M_0 .We denote by $\mathcal{G}_0 = \{up\backslash u\, \mathcal{N}_0, p \in \mathcal{P}(A_0)\}$the set of partial isometries in M_0 normalize$s\, A_0$

The proof becomes very similar to the proof of Theorem (6.2.16). We will only show what the analogue of Lemma (6.2.13) becomes.

Thus, For each finite subset $F \subset \Gamma_1 \cup \Gamma_2\{1\}, n \geq 1$, a non-zero projection $f \in A_0$ and $v \in \mathcal{G}_0$satisfying$vv^* = v^*v \leq f$, we denote by $F_{v,n}$ the set of all elements of the form $x = u_0 \Pi_{i=1}^k v^{\gamma_i} u_i$, where $u_0 \in F \cup \{1\}$, $u_i \in F, \gamma_i = \pm 1$, $1 \leq k \leq n$. We need to show that given any $\varepsilon > 0$ ε > 0, there exists $u \in \mathcal{G}_0$ such that $uu^* = u^*u, \|E_A(x)\|_1 \leq \varepsilon\ \forall x \in F_{u,n}$, and $\tau(uu^* > \tau(f)/4$.

To do this, let $\delta = 2^{-n^2-1}\varepsilon$ and denote $\varepsilon_0 = \delta, \varepsilon_k = 2^{k+1}\varepsilon_{k-1}, k \geq 1$. Note that $\varepsilon_n < \varepsilon$. Let W denote the set of partial isometries $v \in \mathcal{G}_0$ with $vv^* = v^*v \leq f$ such t hat $\|E_A(x)\|_1 \leq \varepsilon_k \tau(\ vv^*), \forall x \in F_{v,k}$ for all $1 \leq k \leq n$, and endow W with the order given by $w_1 \leq w_2$ if $w_1 = w_2 w_1^* w_1$. Noticing that W is well ordered with respect to $\leq$, we let $v \in W$ be a maximal element. Assume that $\tau(\ vv^*) \leq \tau(f)/4$ and note that $p = f - vv^* \in \mathcal{P}(A_0)$ will then satisfy $\tau(\ vv^*)/\tau(p) \leq 1/3$.

If $w \in \mathcal{G}_0$ satisfies $ww^* = w^*w\mathcal{G}_0 \leq p$, then $u = v + w$ belongs to $\mathcal{G}_0$ and satisfies $uu^* = u^*u$. When we develop $u_0 \Pi_{i=1}^k (v + w)^{\gamma_i} u_i$ binomially, we get

$$\left\|E_A(\ u_0 \Pi_{i=1}^k u^{\ \gamma_i} u_i)\right\|_1 \leq \left\|E_A(\ u_0 \Pi_{i=1}^k u^{\ \gamma_i} u_i)\right\|_1 + \sum{}' + \sum{}'',$$

where $\sum''$ is the sum of the L^1-norm of terms that contain at least two occurrences of $w^{\pm 1}$, while $\sum'$ is the sum the L^1-norm of terms containing exactly one occurrence of $w^{\pm 1}$.

To estimate $\sum''$, exactly the same is used in the estimates (2) − (10) in the proof of Lemma (6.2.13), to get that $\sum'' \leq \varepsilon_k \tau(q) - (2k + 4)(\varepsilon_{k-1}/3)\, \tau(q)$,

Note that in order to do that, we only use the properties of the support q of w, namely the fact that given any finite set $Y \subset M \ominus A$ and any $\alpha > 0$, one can take $q \in \mathcal{P}(A_0)$ such that$\in \|qyq\|_1 < \alpha\tau(q), \forall y \in Y$ (by applying to $Q = A_0$ and using the fact that $A_0' \cap M = A$).

Now, in order to estimate $\sum'$, we denote by $\mathcal{U}_q$ the set of partial isometries in $\mathcal{G}_0$ that have left and right support equal to q , which we view as a subgroup of unitaries in qM_0q. Notice that $\mathcal{U}_q$ generate qM_0q and that $M_0 \nprec_M M_0' \cap M$ (because this centralizer is separable and amenable, and by applying Theorem (6.2.8) and Proposition (6.2.14). Thus, given any finite set $Y \subset M$ and any $\alpha > 0$, there exists unitary elements $w \in \mathcal{U}_q$ such that $\|E_A(y_1 w y_2)\|_1 < \alpha\tau(q), \forall y_1, y_2 \in Y$.

Then the same estimates as the ones in (11) − (14) in the proof of Lemma (6.2.13), show that by $u = v + w \in W$, contradicting the maximality of by v. Thus, we do have indeed $\tau(vv^*) > \tau(f)/4$. With this technical fact in hand, the rest of the proof proceeds exactly as the proof of Theorem (6.2.16).

Theorem (6.2.21)[260]: Assume $Q \subset M$ is either in the class Q_u, or Q_b. If $X \subset M \ominus (Q' \cap M)$ is a separable subspace, then there exists a diffuse abelian von Neumann subalgebra $A \subset Q$ such that A is free independent to X, relative to $Q' \cap M$, more precisely $E_{Q' \cap M}(x_0 \prod_{\epsilon=0}^{1+\epsilon} a_{1+\epsilon} x_{1+\epsilon}) = 0$, for all $\epsilon \geq 0$ and all $x_{1+\epsilon} \in X, \epsilon \geq 0$, $x_0, x_{2+2\epsilon} \in X \cup \{1\}$, $a_{1+\epsilon} \in A \ominus \mathbb{C}, \epsilon \geq 0$.

Proof: We construct recursively a sequence of partial isometries $v_1, v_2, \dots \in Q$ such that

(i) $v_{2+\epsilon} v_{1+\epsilon}^* v_{1+\epsilon} = v_{1+\epsilon}$, $v_{1+\epsilon} v_{1+\epsilon}^* = v_{1+\epsilon}^* v_{1+\epsilon}$ and $\tau(v_{1+\epsilon} v_{1+\epsilon}^*) \geq 1 - \frac{1}{2(1+\epsilon)}$, $\forall \epsilon \geq 0$.

(ii) $E_{Q' \cap M}(x) = 0, \forall x \in X_{v_{1+\epsilon}}^{1+\epsilon}, \epsilon \geq 0$.

Assume we have constructed $v_{1+\epsilon}$ for $\epsilon = 0, \dots, m-1$. If v_m is a unitary element, then we let $v_{1+\epsilon} = v_m$ for all $1 + \epsilon \geq m$. If v_m is not a unitary element, then let $f = 1 - v_m^* v_m \in Q$. Note that $E_{Q' \cap M}(x') = 0$, for all $x' \in X' \overset{\text{def}}{=} \cup_{1+\epsilon} X_{v_m}^{1+\epsilon}$. Thus, if we apply Lemma (6.2.13) to $Q \subset M$, the projection $f \in Q$ and the countable set $X' \subset M \ominus (Q' \cap M)$, then we get a partial isometry $u \in fQf$, with $uu^* = u^*u$ satisfying $\tau(uu^*) \geq \tau(f)/2$ and $E_{Q' \cap M}(x) = 0$ for all $x \in \cup_{1+\epsilon} (X')_u^{1+\epsilon}$. But then $v_{m+1} = v_m + u$ will satisfy both (i) and (ii) for $\epsilon = m$.

It follows now from (i) that the sequence $v_{1+\epsilon}$ converges in the norm $\| \|_2$ to a unitary element $v \in Q$, which due to (ii) will satisfy the condition $E_{Q' \cap M}(x), \forall x \in \cup_{1+2\epsilon} X_v^{1+2\epsilon}$. Now, since Q is a II_1 von Neumann algebra, Q contains a copy of the hyperfinite II_1 factor, which in turn contains a Haar unitary $u_0 \in R$. But then $u = vu_0v^*$ clearly satisfies the conditions required in part (a) of Theorem (6.2.16).

List of Symbols

Symbol		Page
$\otimes$:	tensor product	8
sup :	supremum	12
LUR:	locally uniformly rotund	43
SLD:	Small local d-diameter	43
diam:	diameter	46
WCD:	weakly countably determined	50
WCG:	weakly compactly generated	51
Co:	convex	51
supp:	support	63
Aut:	Automorphism	65
inf :	infimum	67
Re:	Real	68
$\oplus$:	orthogonal sum	72
L^2:	Hilbert space	78
$\ominus$:	Direct difference	78
$e^{\infty}(N, M)$:	von Neumann algebra	79
Mod:	modulo	100
det:	determinant	100
mat:	matrix	100
min:	minimum	103
max :	maximum	107
diag:	diagonal	117
tr:	trace	123
GL_2:	Hilbert-Lie group	136
ind:	Index	138
Cyl:	cylindrical	167
dom:	domain	171
seq°:	sequential	173
Ba:	Baire	174
ULT:	Ultra	174
clop:	clopen	174
OUT:	Outer	194
dim:	dimension	196
incl:	inclusion	198
rep:	representation	202
MASA:	maximal abelian sub algebras	214
AFD:	approximately finite dimensional	217
IC:	infinite conjugacy	218

References

[1] G.W. Mackey, Imprimitivity for representations of locally compact groups. Proc. Nat. Acad. Sci. U.S.U. 35 (1949) 537–545.

[2] Cartan, H., Ann. ccole normale superieure, 64, 59-77 (1947), Theorem 4.

[3] Dieudonne, J., Bull. Soc. math. France, 71, 27-45 (1943).

[4] Dickson, L. E., Linear Groups, Leipzig, 1901.

[5] Van der Waerden, B. L., Gruppen von linearen Transformationen, Berlin, 1935, p. 7.

[6] Iwasawa, M., Proc. Imp. Acad. Japan, 17, 57 (1941).

[7] GELFAND, I. and NEUMARK, M., Unitaryr epresentationsof theg roupo f linear trans-formations of the straight line, C. R. (Doklady) Acad. Sci. URSS (NS) vol. 55 (1947), pp. 567-570.

[8] WIGNER, E., On unitaryr epresentationsof the inhomogeneousLo rentzg roup,A nn. of Math., vol. 40 (1939), pp. 149-204.

[9] GODEMENT, R., Sur la transformationde Fourierd ans les groupsd iscrets,C . R. Acad. Sci. Paris, vol. 228 (1949), pp. 1627-1628.

[10] Weil, A., L'integration dans les groupes topologiques et ses applications, Paris (1940), pp. 82, 83.

[11] G.W. Mackey, Induced representations of locally compact groups. I, Ann. of Math. (2) 55 (1952) 101–139.

[12] AMBROSE, W., Spectral resolution of groups of unitary operators, Duke Math. J., vol. 11 (1944), pp. 589-595.

[13] DIEUDONNt, J., Sur le theoremed e Lebesgue-Nikodym (III), Ann. Univ. Grenoble (NS) vol. 23 (1948), pp. 25-53.

[14] FEDVRER, H. and MORSF,, A. P., Some propertiesof measurablef unctions, Bull. Amer. Math. Soc., vol. 49 (1943), pp. 270-277.

[15] GELFAND, I. and NEUMARK, M., Unitaryr epresentationsof theg roupo f linear trans-formations of the straight line, C. R. (Doklady) Acad. Sci. URSS (NS) vol. 55 (1947), pp. 567-570.

[16] GELFAND, I., and NEUMARK, M., Unitary representations of the Lorentz group, Bull. Acad. Sci. URSS. Ser. Math. vol. 11 (1947), pp. 411-504.

[17] GELFAND, I., AND NEUMARK, M., Unitary representations of semi simple Lie groups I, Rec. Math. (Mat. Sbornik) (NS) vol. 21 (1947), pp. 405-434.

[18] GELFAND I., and NEUMARK, M., On the connection between the representations of a complex semis imple Lie group and thoseofits maximal compacts ubgroup, C. R. (Doklady) Acad. Sci. USSR (NS) vol. 63 (1948), pp. 225-228.

[19] GELFAND, I., and NEUMARK, M., The relation between the unitary representations of the complex uni modular group and its unitary subgroup, Bull. Acad. Sci. URSS. Ser. Math. Vol. 14 (1950) pp. 239-260.

[20] GODEMENT, R., Sur une generalization d 'un theorbmede Stone,C . R. Acad. Sci. Paris, vol. 218 (1944), pp. 901-903.

[21] GODEMENT, R., Theorem general ed es sommes continuesd 'espacesd e Banach, C. R. Acad. Sci. Paris, vol. 228 (1949), pp. 1321-1323.

[22] GODEMENT, R., Sur la theoried es representationsun itaires, A nn. ofM ath. vol. 53 (1951), pp. 68-124.

[23] HALMOS, P. R., Thed ecompositionof m easures, Duke Math. J.,v ol. 8 (1941),p p. 386-392.
[24] HALMOS, P. R., On a theoremof D ieudonn4, P roc. Nat. Acad. Sci. U. S. A., vol. 35 (1949), pp. 38-42.
[25] HALMOS, P. A., Measure Theory, New York, 1950.
[26] KURATOWSKI, C., Topologie I, Warsaw, 1933.
[27] MACKEY, G. W., On a theorem of Stone and von Neumann, D uke Math. J. vol. 16 (1949), pp. 313-326.
[28] MACKEY, G. W., Imprimitivity for representations of locally compact groups I, Proc. Nat. Acad. Sci. U. S. A., vol. 35 (1949), pp. 537-545.
[29] MACKEY, G. W., Imprimitivitepo ur les representations des groups localement compacts III. Produitsd e Kroneckeret n ombres d'entrelacementfor ts.,C . R. Acad. Sci. Paris, vol. 230 (1950), pp. 808-809.
[30] MACKEY, G. W., Induced representations of groups, A mer. J. Math. vol. 73 (1951) pp. 576-592.
[31] MAUTNER, F. I., The completeness of the irreducible unitary representations of a locally compact group, P roc. Nat. Acad. Sci. U. S. A., vol. 34 (1948), pp. 52-54.
[32] MAUTNER, F. I., Unitary representations of locally compact groups I, Ann. Of Math. vol. 51 (1950), pp. 1-25.
[33] MAUTNER, F. I., Unitary representations of locally compact groups II, Ann. of Math. vol. 51 (1950), pp. 528-556.
[34] VON NEUMANN, J. Zur Operator en method einder KlassischenM echanik, Ann. of Math. vol. 33 (1932), pp. 789-791.
[35] VON NEUMANN, J., On the uniqueness of invariant Lebesgue measure, a bstract, Bull. Amer. Math. Soc. vol. 42 (1936) p. 343.
[36] VON NEUMANN J., On Ring sof operators, Reductiont heory, Ann. of Math. vol. 50 (1949X, 401-485.
[37] NEUMARK, M., Positive definite operator functions on a commutative group, Bull. Acad. Sci. URSS. Ser. Math. vol. 7 (1943) pp. 237-244.
[38] ROHLIN, V. A., On the fundamental ideas of measure theory, Rec. Math. (Mat. Sbornik) (NS), vol. 25 (1949), pp. 107-150.
[39] SEITZ, F., On the reduction of space groups, Ann. of Math., vol. 37 (1936), pp. 17-28.
[40] STONE, M. H., On one parameter unitary groups in Hilberts pace, Ann. of Math. vol. 33 (1932), pp. 643-648.
[41] URSELL, H. D., Some methods of proving measurability, Fund. Math. vol. 32 (1939) pp. 311-330.
[42] WEIL, A., Lintegrations dans les groups topologi queset ses applications, A ctualites Scientifiquee t Industrielles,n o. 869, Paris, 1938.
[43] Z. Ciesielski, A. Pelczyński, W. Żelazko, STUDIA MATHEMATICA, Vols. 132-137 (18 issues) 1999.
[44] G. A. Alexandrov, One generazation of the Kadec propertij, preprint, 1993.
[45] BESSAGA, C. AND PEŁCZY ´NSKI, A., (1975). Selected Topics in Infinite Dimensional Topology, Polish Scientific Publishers, Warsaw.

[46] B. Cascales and G. Vera, Topologies weaker than the weak topology of a Banach space, J. Math, Anal. App]. 182 (1094), 41-68.
[47] R. Devl1e, G. Godefroy and V. Zizier, Smoothness and Renorinings in Banach Spaces, Pitinan Monographs Surveys 64, Longinan Sd. Tech., 1993.
[48] G. A. Edgar, Meas.årnhility in a .Banach space, Indiana Univ. Math, J. 26 (1977), 663—677,
[49] Measttrrsbilitij in a Bcrnach space LI, ibid. 28 (1979), 559—579.
[50] A long James space, in: Measure Theory (Oberwolfach 1979), Lecture Notes zn Math. 794, Springer, 1980, 31—37.
[51] M. Fabian, On a dual locally unifonnly rotund norm on a duo) Vašdk space, Studia. Math. 101 (1991), 69—81.
[52] M. Fabian and G. Godefroy, The dual of etrcry Asplund space admits a projectional resolution of the identity, ibid. 91 (1988), 141—151.
[53] R. Hansell, ‘Descriptive sets and the topology of non-separable Banach spaces’, Serdica Math. J. 27 (2001) 1–66.
[54] R. G. Haydon, ‘Trees in renorming theory’, Proc. London Math. Soc. 78 (1999) 541–584.
[55] Baire frees, bad norms and the Namioka property, preprint.
[56] R. G. Haydon, J. E. Jayne, I. Namioka and C. A. Rogers, ‘Continuous functions on totally ordered spaces that are compact in their order topologies’, J. Funct. Anal. 178 (2000) 23–63.
[57] J. E. Jayne, f. Nainioka and C. A. Rogers, Nor-sn fragmented weak* compact sets, Couvct. Mat. 41 (1990), 133—163.
[58] A-fragrncntcible Banach spaces, Mathematilca 39 (1992), 161—188 and 197--215.
[59] Topological properties of Bana.ch spaces, Proc. London Math. ‘Soc. (3) 66 (1993), 651—672.
[60] Continuous fisnclwns on products 01 compact Flausdoiff spaces, to appear.
[61] C. L anclen, Tkéorgc de l’indice et problèmes de renormage en géométrie des espaces de Banach, thèse, Paris, 1992.
[62] A. Moltô, J, Orihuela and S. Troyanski, Locally u,nfortn rotund renormrng and fragmentability, Proc. London Math. Soc. (3) 75 (1997), 619—640.
[63] A. Moltó, J. Orihuela, S. Troyaneki and M. Valdivia1 On weakly locally uniformly rotund Ba’w.ch spaces, to appear.
[64] I. Namioka, Radon-Nikodym compoæt spaces and fragmeritabilittj, Mathematika, 34 (1989) 258—281.
[65] L. Onci na, Borel set. and o--fragmenta&ili.Ly of a Banach space, Mast.er Degree Thesis at University College London, 1996.
[66] M. Raja, On topology and renormtng of a Banach space, C. R. Aca.d. Bulgare Sel., to appear.
[67] M. Talagranci, Sur une conjecture de II. H. Corson, Bull. Sci. Math. 99 (1975), 211-212.
[68] Sur la structure borthenne des espaces analytiques, ibid. 101 (1977), 415-422.
[69] M. Talagrand, ‘Comparaison des boreliens d’un espace de Banach pour les topologies fortes et faibles’, Indiana Univ. Math. J. 27 (1978) 1001–1004.

[70] M. Valdivia, Projective resolutions of identitj in C(K) spaces, Arch, Math. (Basel) 54 (1990), 493—498.
[71] L. Vašk, On one generalrzation of weakly compactly gewrated space, Studla. Math. 70 (1981), 11—19.
[72] BURKE, D. K. AND POL, R., (2003). On Borel sets in function spaces with the weak topology, J. Lond. Math. Soc. (2), 68, 3, 725–738.
[73] M. R. Burke, ‘Borel measurability of separately continuous functions’, Topology Appl. 129 (2003) 29–65.
[74] R. Deville, G. Godefroy and V. Zizler, ‘Smoothness and renormings in Banach spaces’, Pitman Monographs and Surveys in Pure and Applied Mathematics 64 (Longman, Harlow, 1993).
[75] A. Dow, ‘Saturated Boolean algebras and their Stone spaces’, Topology Appl. 21 (1985) 193–207.
[76] D. H. Fremlin, ‘Measure-additive coverings and measurable selectors’, Dissertationes Math. 260 (1987).
[77] L. Gillman and M. Jerrison, Rings of continuous functions (Van Nostrand, New York, 1960).
[78] R. G. Haydon, ‘Some problems about scattered spaces’, S´eminaire d’Initiation `a l’Analyse (1989/90), Publications Math´ematiques de l’Universit´e Pierre et Marie Curie 95 (Universit´e Paris VI, Paris).
[79] R. G. Haydon and V. Zizler, ‘A new space with no locally uniformly rotund renorming’, Canad. Math. Bull. 32 (1989) 122–128.
[80] J. E. Jayne, I. Namioka and C. A. Rogers, ‘σ-fragmentable Banach spaces’, Mathematika 39 (1992) 197–215.
[81] J. E. Jayne, I. Namioka and C. A. Rogers, ‘Fragmentability and σ-fragmentability’, Fund. Math. 143 (1993) 207–220.
[82] A. S. Kechris, Classical descriptive set theory (Springer, New York, 1995).
[83] P. S. Kenderov, I. S. Kortezov and W. B. Moors, ‘Continuity points of quasi-continuous mappings’, Topology Appl. 109 (2001) 321–346.
[84] KENDEROV, P. S.; KORTEZOV, I. S. AND MOORS, W. B., (2006). Norm continuity of weakly continuous mappings into Banach spaces, Topology Appl., 153, 2745–2759.
[85] S. Mercourakis and S. Negrepontis, ‘Banach spaces and topology II’, Recent progress in general topology (ed. M. Huˇsek and J. van Mill, Elsevier, 1992).
[86] I. Namioka and R. Pol, ‘Mappings of Baire spaces into function spaces and Kadeˇc renorming’, Israel J. Math. 78 (1992), 1–20.
[87] S. Negrepontis, ‘Banach spaces and topology’, Handbook of set-theoretic topology (ed. K. Kunen and J. E. Vaughan, Elsevier, 1984).
[88] L. Oncina, ‘The JNR property and the Borel structure of a Banach space’, Serdica Math. J. 26 (2000) 13–32.
[89] H. P. Rosenthal, ‘On relatively disjoint families of measures, with some applications to Banach space theory’, Studia Math. 37 (1970) 13–36.
[90] M. Talagrand, ‘Pettis integral and measure theory’, Memoirs of the American Mathematical Society 307 51 (American Mathematical Society, Providence, RI, 1984).

[91] E. K. van Douwen, ‘A technique for constructing honest locally compact submetrizable examples’, Topology Appl. 47 (1992) 179–201.
[92] S. Popa: On a problem of R.V. Kadison on maximal abelian *-subalgebras in factors, Invent. Math., 65 (1981), 269-281.
[93] Anderson, J., Bunce, J.: Stone-Weierstrass theorems for separable C*-algebras. J. Operator Theory 6, 363-374 (1981).
[94] Arveson, W.: Analyticity in operator algebras. Amer. J. Math. 89(3), 578-641 (1967).
[95] Connes, A.: Almost periodic states and factors of type III 1. J. Functional Analysis 16(4), 415- 445 (1974).
[96] Connes, A.: Classification of injective factors. Ann. of Math. 104, 73 115 (1976).
[97] J. Dixmier: Sous-anneaux abeliens maximaux dans les facteurs de type fini, Ann. of Math. 59 (1954), 279-286.
[98] Fuglede, B., Kadison, R. V.: On a conjecture of Murray and von Neumann. Proc. Math. Acad. Sci. USA 37, 420-425 (1951).
[99] Jones, V., Popa, S.: Some properties of MASA's in factors, preprint.
[100] Kadison, R.V.: Problems on yon Neumann algebras. Baton Rouge Conference, 1967 (unpublished).
[101] McDuff, D.: Central sequences and the hyperfinite factor. Proc. London Math. Soc. 21, 443- 461 (1970).
[102] StrS.til~, S., Zsido, L.: Lectures in yon Neumann algebras. Editura Academiei, Bucure~ti and Tunbridge Wells. Abacus Press 1979.
[103] S. STRATILA, “Modular Theory in Operator Algebras,” Editura Academiei/Abacus Press, Bucuresti/Tunbridge Wells, 1981.
[104] S. Popa: Maximal injective subalgebras in factors associated with free groups, Advances in Math., 50 (1983), 27-48.
[105] W. M. CHING, Free products of von Neumann algebras, Trans. Amer. Math. Sot. 178 (1973), 147-163.
[106] M. CHODA, The crossed product of a full II,-factor by a group of outer automorphisms, Math. Japan. 23 (4) (1978), 385-391.
[107] M. CHODA, Inner amenability and fullness, to appear.
[108] A. CONNES, Outer conjugacy classes of automorphisms of factors, Ann. Ecole Norm. Sup. 8 (1975), 383-419.
[109] A. CONNES, J. FELDMAN, AND B. WEISS, An amenable equivalence relation is generated by a single transformation, Ergod. Theory and Dynam. Systems 1(1981), 43 l-450.
[110] A. CONNES AND E. ST~RMER, Entropy of automorphisms of II, von Neumann algebras, Acta Math. 134 (1975), 289-306.
[111] J. DIXMIER, Quelques propribtes des suites centrales dans les facteurs de type II,, Invent. Math. 7 (1969), 215-225.
[112] H. DYE, On groups of measure preserving transformations, I, II, Amer. J. Math. 81 (1959), 119-159; 85 (1963), 551-576.
[113] V. F. R. JONES, A converse to Ocneanu’s theorem, J. Oper. Theory 10 (1983), in press.

[114] V. F. R. JONES, Central sequences in crossed products of full factors, Duke Math. J. 49 (1982), 29-34.
[115] R. V. KADISON, Normalcy in operator algebras, Duke Math. J. 29 (1962), 459-464.
[116] R. V. KADISON, Problems on von Neumann algebras, Baton Rouge Conference, unpublished.
[117] R. LYNDON AND P. SCHUPP, "Combinatorial Group Theory," Springer-Verlag, Berlin/Heidelberg/New York, 1977.
[118] F. MURRAY AND J. VON NEUMANN, Rings of operators, IV, Ann. of Math. 44 (1943), 716-808.
[119] A. OCNEANU, "Actions of Discrete Amenable Groups on Factors," Thesis, Warwick University.
[120] J. PHILIPS, Automorphisms of full II,-factors with applications to factors of type III, Duke Math. J. 43 (1976), 375-385.
[121] S. POPA, Orthogonal pairs of subalgebras in finite von Neumann algebras, J. Oper. Theory 9 (1983), 253-268.
[122] K. SCHMIDT, Amenability, Kazhdan's property T, strong ergodicity and invariant means for ergodic group actions, Ergod. Theory and Dynam. Systems 1(1981), 223-236.
[123] S. Albeverio and A. Kosyak, Quasiregular representations of the infinite-dimensional nilpotent group, J. Funct. Anal. 236 (2006) 634-681.
[124] S. Albeverio, R. Höegh-Krohn, The energy representation of Sobolev–Lie group, Compos. Math. 36 (1978) 37–52.
[125] S. Albeverio, A. Kosyak, Quasiregular representations of the infinite-dimensional Borel group, J. Funct. Anal. 218 (2) (2005) 445–474.
[126] S. Albeverio, A. Kosyak, Group action, quasi-invariant measures and quasiregular representations of the infinitedimensional nilpotent group, Contemp. Math. 385 (2005) 259–280.
[127] S. Albeverio, R. Höegh-Krohn, D. Testard, Irreducibility and reducibility for the energy representation of a group of mapping of a Riemannian manifold into a compact Lie group, J. Funct. Anal. 41 (1981) 378–396.
[128] S. Albeverio, R. Höegh-Krohn, D. Testard, A. Vershik, Factorial representations of path groups, J. Funct. Anal. 51 (1983) 115–131.
[129] S. Albeverio, R. Höegh-Krohn, J. Marion, D. Testard, B. Torrésani, Noncommutative Distributions, Unitary Representation of Gauge Groups and Algebras, Monogr. Textbooks Pure Appl. Math., vol. 175, Dekker, New York, 1993.
[130] E.F. Beckenbach, R. Bellmann, Inequalities, Springer, Berlin, 1961.
[131] J. Dixmier, Les C∗-algèbres et leur représentation, Gautier–Villars, Paris, 1969.
[132] J. Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien, second ed., Gauthier–Villars, Paris, 1969.
[133] R.F. Gantmacher, Matrizenrechnung, Teil 1, VEB, Berlin, 1958.
[134] I.M. Gel'fand, A.M. Vershik, M.I. Graev, Representations of SL2(R), where R is a ring of functions, Uspekhi Mat. Nauk 28 (1973) 83–128.
[135] R.A. Horn, C.R. Jonson, Matrix Analysis, Cambridge Univ. Press, Cambridge, 1989.

[136] R.A. Horn, C.R. Jonson, Topics in Matrix Analysis, Cambridge Univ. Press, Cambridge, 1991.
[137] R.S. Ismagilov, Representations of the group of smooth mappings in a compact Lie group, Funkts. Anal. I Priložhen. 15 (2) (1981) 73–74 (in Russian).
[138] S. Kakutani, On equivalence of infinite product measures, Ann. of Math. 4 (1948) 214–224.
[139] A. A. Kirillov, Elements of the Theory of Representations, Grundlehren Math. Wiss., vol. 220, Springer, Berlin, 1976 (translated from the Russian).
[140] A. A. Kirillov, Introduction to the theory of representations and noncommutative harmonic analysis, in: Representation Theory and Noncommutative Harmonic Analysis, I, in: Encyclopaedia Math. Sci., vol. 22, Springer, Berlin, 1994, pp. 1–156.
[141] A.V. Kosyak, Irreducibility criterion for regular Gaussian representations of group of finite upper triangular matrices, Funkts. Anal. i Priložhen. 24 (3) (1990) 82–83 (in Russian); transl. in: Funct. Anal. Appl. 24 (3) (1990) 243–245.
[142] A.V. Kosyak, Criteria for irreducibility and equivalence of regular Gaussian representations of group of finite upper triangular matrices of infinite order, Selecta Math. Soviet. 11 (1992) 241–291.
[143] A.V. Kosyak, Irreducible regular Gaussian representations of the group of the interval and the circle diffeomorphisms, J. Funct. Anal. 125 (1994) 493–547.
[144] A.V. Kosyak, Regular representations of the group of finite upper-triangular matrices, corresponding to product measures, and criteria for their irreducibility, Methods Funct. Anal. Topology 6 (2000) 43–65.
[145] A.V. Kosyak, The generalized Ismagilov conjecture for the group BN 0. I, Methods Funct. Anal. Topology 8 (2) (2002) 33–49.
[146] A.V. Kosyak, The generalized Ismagilov conjecture for the group BN 0. II, Methods Funct. Anal. Topology 8 (3) (2002) 27–45.
[147] A.V. Kosyak, A criterion for irreducibility of quasiregular representations of the group of finite upper-triangular matrices, Funkts. Anal. i Priložhen. 37 (1) (2003) 78–81 (in Russian); transl. in: Funct. Anal. Appl. 37 (1) (2003) 65–68.
[148] A.V. Kosyak, Quasi-invariant measures and irreducible representations of the inductive limit of the special linear groups, Funkts. Anal. i Priložhen. 38 (1) (2004) 82–84 (in Russian); transl. in: Funct. Anal. Appl. 38 (1) (2004) 67–68.
[149] H.H. Kuo, Gaussian Measures in Banach Spaces, Lecture Notes in Math., vol. 463, Springer, Berlin, 1975.
[150] S.M. Malamud, A converse to the Jensen inequality, its matrix extensions and inequalities for minors and eigenvalues, Linear Algebra Appl. 322 (1–3) (2001) 19–41.
[151] M.P. Malliavin, Naturality of quasi-invariance of some measures, in: A.B. Cruzeiro, J.-C. Zambrini (Eds.), Stochastic Analysis and Applications, Lisbon, 1989, in: Progress in Probability, vol. 26, Birkhäuser Boston, Boston, MA, 1991, pp. 144–154.
[152] M.P. Malliavin, Probability and geometry, in: Taniguchi Conference on Mathematics, Nara’98, in: Adv. Stud. Pure Math., vol. 31, Math. Soc. Japan, Tokyo, 2001, pp. 179–209.
[153] M.P. Malliavin, P. Malliavin, Measures quasi invariant es sur certain groupes de dimension infinie, C. R. Acad. Sci. Paris Sér. I Math. 311 (1990) 765–768.

[154] M.P. Malliavin, P. Malliavin, Integration on loop groups, I, J. Funct. Anal. 93 (1990) 207–237.
[155] M.P. Malliavin, P. Malliavin, Integration on loop groups, III: Asymptotic Peter–Weyl orthogonality, J. Funct. Anal. 108 (1992) 13–46.
[156] N.I. Nessonov, Examples of factor-representations of the group GL(∞), in: Mathematical Physics, Functional Analysis, Naukova Dumka, Kiev, 1986, pp. 48–52 (in Russian).
[157] M. Reed, B. Simon, Methods of Modern Mathematical Physics, vol. I, Academic Press, New York, 1972.
[158] E.T. Shavgulidze, Distributions on infinite-dimensional spaces and second quantization in string theories, II, in: V International Vilnius Conference on Probability Theory and Math. Statistics, Abstracts of Comm., Vilnius, June 26–July 1, 1989, pp. 359–360.
[159] G.E. Shilov, Fan Dik Tun', Integral, Measure, and Derivative on Linear Spaces, Nauka,Moscow, 1967 (in Russian).
[160] D.P. Zhelobenko, A.I. Shtern, Representations of Lie Groups, Nauka, Moscow, 1983.
[161] A. V. KOSYAK, INDUCED REPRESENTATIONS OF INFINITE-DIMENSIONAL GROUPS, arXiv: 1207.0076v1 [math. RT]30 Jun 2012.
[162] J. Dixmier, Les alg`ebres d'op´erateurs dans l'espace hilbertien, 2nd Edition, Gauthier-Villars, Paris, 1969.
[163] A.A. Kirillov, Unitary representations of nilpotent Lie groups, Usp. Mat. Nauk. (1962) 17, No 4, 57-110.
[164] A.A. Kirillov, Representations of infinite-dimensional unitary groups, Dokl. Akad. Nauk SSSR, 212 (1973) No. 2, 288-290, English transl.: Sov. Math. Dokl. 14 (1974) 1355-1358.
[165] A.V. Kosyak, Extension of unitary representations of inductive limits of finite–dimensional Lie groups, Rep. Math. Phys. 26/2 (1988) 129–148.
[166] A.V. Kosyak Inversion-quasi-invariant Gaussian measures on the group of infinite-order uppertriangular matrices. Funct. Anal. i Priloz. 34 issue 1, (2000) 86–90.
[167] G.I. Ol'shanski˘ı, Unitary representations of infinite-dimensional pairs (G,K) and the formalism of R. Howe, in: Representation of Lie Groups and Related Topics, A.M. Vershik and D.P. Zhelobenko (Eds.), Advance Studies Contemporary Mathematics, vol. 7, Gordon and Breach, New York, 1990, pp. 269–463.
[168] A. Weil, L'int´egration dans les groupes topologoque et ses application 2e ed., Hermann, Paris, 1953.
[169] Xia-Dao-Xing. Measures and Integration in Infinite-Dimensional Spaces, Academic Press, New York/London, 1978.
[170] W. Marciszewski and R. Pol, On some problems concerning Borel structures in function spaces, RACSAM Rev. R. Acad. Cienc. Exactas F´ıs. Nat. Ser. A Mat., 104 (2010), 327–335.
[171] ALBIAC, F. AND KALTON, N. J., (2006). Topics in Banach Space Theory, Springer.
[172] DODOS, P., (2006). Codings of separable compact subsets of the first Baire class, Ann. Pure Appl. Logic, 142, 425–441.
[173] DOW, A. AND HART, K. P., (2000). The measure algebra does not always embed, Fund. Math., 163, 163–176.

[174] EDGAR, G. A., (1977). Measurability in a Banach space, Indiana Univ. Math. J., 26, 4, 663–677.
[175] EDGAR, G. A., (1979). Measurability in a Banach space. II, Indiana Univ. Math. J., 28, 4, 559–579.
[176] FREMLIN, D. H., (1980). Borel sets in non-separable Banach spaces, Hokkaido Math. J., 9, 179–183.
[177] FREMLIN, D. H., (2003). Measure Theory, Vol. 4: Topological Measure Theory, Torres Fremlin.
[178] GODEFROY, G., (1980). Compacts de Rosenthal, Pacific J. Math., 91, 293–306.
[179] HALMOS, P. R., (1963). Lectures on Boolean Algebras, Van Nostrand.
[180] HAYDON, R.; MOLT´O, A. AND ORIHUELA, J., (2007). Spaces of functions with countably many discontinuities, Israel J. Math., 158, 19–39.
[181] KURATOWSKI, K., (1966). Topology I, Academic Press and PWN, New York and London.
[182] MARCISZEWSKI, W., (1989). On a classification of pointwise compact sets of the first Baire class functions, Fund. Math., 133, 195–209.
[183] MARCISZEWSKI, W., (2008). Modifications of the double arrow space and related Banach spaces C(K), Studia Math., 184, 249–262.
[184] MARCISZEWSKI, W. AND POL, R., (2009). On Banach spaces whose norm-open sets are Fσ-sets in the weak topology, J. Math. Anal. Appl., 350, 708–722.
[185] MARCISZEWSKI, W. AND POL, R., (2010). On Borel almost disjoint families, preprint.
[186] ORIHUELA, J., (2007). Topological open problems in the geometry of Banach spaces, Extracta Math., 22, 197–213.
[187] OSTASZEWSKI, A. J., (1974). A characterization of compact, separable, ordered spaces, J. Lond. Math. Soc. (2), 7, 758–760.
[188] OXTOBY, J. C., (1960/1961). Spaces that admit a category measure, J. Reine Angew. Math., 205, 156–170.
[189] POL, R., (1986). Note on compact sets of first Baire class functions, Proc. Amer. Math. Soc., 96, 152–154.
[190] RAJA, M., (1999). Kadec norms and Borel sets in a Banach space, Studia Math., 136, 1–16.
[191] SAKAI, S., (1971). C∗-algebras and W∗-algebras, Springer.
[192] TODORCEVIC, S., (2005). Representing trees as relatively compact subsets of the first Baire class, Bull. Cl. Sci. Math. Nat. Sci. Math., 30, 29–45.
[193] VAKHANIA, N. N.; TARIELADZE, V. I. AND CHOBANYAN, S. A., (1991). Probability Distributions on Banach Spaces, Kluwer.
[194] A. Avil´es, G. Plebanek, and J. Rodr´ıguez, On Baire measurability in spaces of continuous functions, J. Math. Anal. Appl. 398 (2013), 230–238.
[195] A. Blass, R. Frankiewicz, G. Plebanek, C. Ryll-Nardzewski, A note on extensions of asymptotic density, Proc. Amer. Math. Soc. 129 (2001), no. 11, 3313–3320.
[196] P. Borodulin-Nadzieja and G. Plebanek, On sequential properties of Banach spaces, spaces of measures and densities, Czechoslovak Math. J. 60(135) (2010), no. 2, 381–399.

[197] P. Borodulin-Nadzieja and O. Selim, Sequential closure in the space of measures, preprint, arXiv: 1203.4270v1.
[198] J. Diestel and J.J. Uhl, Jr., Vector measures, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977.
[199] M. Dˇzamonja, G. Plebanek, Strictly positive measures on Boolean algebras, J. Symbolic Logic 73 (2008), no. 4, 1416–1432.
[200] M. Fabian, P. Habala, P. H´ajek, V. Montesinos Santaluc´ıa, J. Pelant, and V. Zizler, Functional analysis and infinite-dimensional geometry, CMS Books in Mathematics/Ouvrages deMath´ematiques de la SMC, vol. 8, Springer-Verlag, New York, 2001.
[201] D.H. Fremlin, Measure theory. Vol. 4: Topological measure spaces, Torres Fremlin, Colchester, 2006.
[202] D.H. Fremlin and M. Talagrand, A decomposition theorem for additive set-functions, with applications to Pettis integrals and ergodic means, Math. Z. 168 (1979), no. 2, 117–142.
[203] K. Kunen, A compact L-space under CH, Topology Appl. 12 (1981), no. 3, 283–287.
[204] Z. Lipecki, Extensions of additive set functions with values in a topological group, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys. 22 (1974), 19–27.
[205] A. Mekler, Finitely additive measures on N and the additive property, Proc. Amer. Math. Soc. 92 (1984), no. 3, 439–444.
[206] S. Mercourakis, Some remarks on countably determined measures and uniform distribution of sequences, Monatsh. Math. 121 (1996), no. 1-2, 79–111.
[207] H. Niederreiter, On the existence of uniformly distributed sequences in compact spaces, Composition Math. 25 (1972), 93–99.
[208] D. Plachky, Extremal and monogenic additive set functions, Proc. Amer. Math. Soc. 54 (1976), 193–196.
[209] G. Plebanek, Nonseparable Radon measures and small compact spaces, Fund. Math. 153 (1997), no. 1, 25–40.
[210] J. Rodr´ıguez and G. Vera, Uniqueness of measure extensions in Banach spaces, Studia Math. 175 (2006), no. 2, 139–155.
[211] Z. Semadeni, Banach spaces of continuous functions, MonografieMatematyczne, Tom 55, PWN- Polish Scientific Publishers, Warsaw, 1971.
[212] S. Shelah, Proper forcing, Lecture Notes in Mathematics, 940, Springer-Verlag, Berlin, 1982.
[213] WITOLD MARCISZEWSKI AND GRZEGORZ PLEBANEK, ON BOREL STRUCTURES IN THE BANACH SPACE C(_!), arXiv: 1309.1908v1 [math. FA]7 Sep 2013.
[214] A. Avil´es, G. Plebanek, J. Rodr´ıguez, Measurability in C(2_) and Kunen cardinals, Israel J. Math. 195 (2013), 1–30.
[215] T. Bartoszyski and H. Judah, Set theory. On the structure of the real line, A K Peters, Ltd., Wellesley, MA, 1995.
[216] A.H. Mekler, Finitely additive measures on N and the additivity property, Proc. Amer. Math. Soc. 92 (1984), 439–444.

[217] J.C. Oxtoby, Measure and category, volume 2 of Graduate Texts in Mathematics. Springer Verlag, New York-Berlin, 1980.
[218] S. Vaes: Factors of type II1 without non-trivial finite index subfactors, Trans. Amer. Math. Soc. 361 (2009), 2587-2606.
[219] D. Bisch & V.F.R. Jones, Algebras associated to intermediate subfactors. Invent. Math. 128 (1997), 89–157.
[220] A. Connes, Noncommutative Geometry, Academic Press, 1994.
[221] A. Connes, A factor of type II1 with countable fundamental group. J. Operator Theory 4 (1980), 151–153.
[222] A. Connes, Periodic automorphisms of the hyperfinite factor of type II1. Acta Sci. Math. 39 (1977), 39–66.
[223] A. Ioana, J. Peterson & S. Popa, Amalgamated free products of w-rigid factors and calculation of their symmetry groups. To appear in Acta Math. math.OA/0505589
[224] V.F.R. Jones, Actions of finite groups on the hyperfinite type II1 factor. Mem. Amer. Math. Soc. 28 (1980), no. 237.
[225] V.F.R. Jones, Index for subfactors. Invent. Math. 72 (1983), 1–25.
[226] S. Popa, Strong rigidity of II1 factors arising from malleable actions of w-rigid groups, Part I. Invent. Math. 165 (2006), 369–408.
[227] S. Popa, Strong rigidity of II1 factors arising from malleable actions of w-rigid groups, Part II. Invent. Math. 165 (2006), 409–451.
[228] S. Popa, On a class of type II1 factors with Betti numbers invariants. Ann. of Math. 163 (2006), 809–899.
[229] S. Popa, Classification of amenable subfactors of type II. Acta Math. 172 (1994), 163–255.
[230] S. Popa, Free-independent sequences in type II1 factors and related problems. Ast´erisque 232 (1995), 187–202.
[231] S. Popa & S. Vaes, Strong rigidity of generalized Bernoulli actions and computations of their symmetry groups. Preprint.
[232] S. Vaes, Rigidity results for Bernoulli shifts and their von Neumann algebras (after Sorin Popa). S´em. Bourbaki, exp. no. 961. Ast´erisque, to appear.
[233] SORIN POPA, INDEPENDENCE PROPERTIES IN SUBALGEBRAS OF ULTRAPRODUCT II1 FACTORS, arXiv: 1308.3982v3 [math. OA]29 Jan 2014.
[234] B. Bekka: Operator-algebraic superrigidity for SL(n, Z), n $\geq$ 3, Invent. Math. 169 (2007), 401-425.
[235] D. Bisch, J.F.R. Jones: Algebras associated with intermediate subfactors, Invent. Math. 128 (1997), 89-157.
[236] N. Brown: Topological dynamical systems assciated with II1 factors, Adv. In Math. 227 (2011), 1665-1699.
[237] N. Brown, K. Dykema, K. Jung: Free entropy dimension in amalgamated free products, Proc. London Math. Soc. 97 (2008), 339-367.
[238] G. Elek, G. Lippner: Sofic equivalence relations, J. Funct. Anal. 258 (2010), 1692-1708.
[239] G. Elek, E. Szabo: Sofic representations of amenable groups, Proc. Amer. Math. Soc. 139 (2011), 4285-4291.

[240] I. Farah, I. Goldbring, B. Hart: Amalgamating Rω-embeddable von Neumann algebras, arXiv: 1301.6816.

[241] I. Farah, B. Hart, D. Sherman: Model theory of operator algebras III: elementary equivalence and II1 factors, arXiv: 1111.0998.

[242] J. Feldman: Nonseparability of certain finite factors, Proc. Amer. Math. Soc. 7 (1956), 23-26.

[243] J. Feldman, C.C. Moore: Ergodic equivalence relations, cohomology, and von Neumann algebras I, II, Trans. Amer. Math. Soc. 234 (1977), 289-324, 325-359.

[244] K. Jung: Amenability, tubularity, and embeddings into Rω, Math. Ann. 338 (2007), 241-248.

[245] F. Murray, J. von Neumann: On rings of operators, Ann. Math. 37 (1936), 116-229.

[246] D. Ornstein, B. Weiss: Ergodic theory of amenable group actions, Bull. Amer. Math. Soc. 2 (1980), 161-164.

[247] N. Ozawa: About the Connes embedding conjecture: algebraic approaches, arXiv: 1212.1700.

[248] L. Paunescu: On sofic actions and equivalence relations, J. Funct. Anal. 261 (2011), 2461-2485.

[249] V. Pestov: Hyperlinear and sofic groups: a brief guide, Bull. Symbolic Logic 14 (2008), no. 4, 449-480.

[250] S. Popa: The commutant modulo the set of compact operators of a von Neumann algebra, J. Funct. Analysis, 712 (1987), 393-408.

[251] S. Popa: Markov traces on universal Jones algebras and subfactors of finite index, Invent. Math. 111 (1993), 375-405.

[252] S. Popa: An axiomatization of the lattice of higher relative commutants of a subfactor, Invent. Math., 120 (1995), 427-445.

[253] S. Popa: The relative Dixmier property for inclusions of von Neumann algebras of finite index, Ann. Sci. Ec. Norm. Sup. 32 (1999), 743-767.

[254] S. Popa: Universal construction of subfactors, J. reine angew. Math., 543 (2002), 39-81.

[255] S. Popa: On the superrigidity of malleable actions with spectral gap, J. Amer. Math. Soc. 21 (2008), 981-1000.

[256] S. Popa: A II1 factor approach to the Kadison-Singer problem, arXiv: 1303.1424, to appear in Comm. Math. Phys.

[257] D. Voiculescu: Symmetries of some reduced free product C∗-algebras, In: “Operator algebras and their connections with topology and ergodic theory”, Lect. Notes in Math. Vol. 1132, 566-588 (1985).

[258] B. Weiss: Sofic groups and dynamical systems, in “Ergodic theory and harmonic analysis”, Mumbai, 1999, Sankhya Ser. A. 62 (2000), 350-359.

[259] F. B. Wright: A reduction for algebras of finite type, Ann. of Math. 60 (1954), 560 - 570.

[260] Shawgy Hussein and Wijdan Mohammed Khalfa, Infinite-Dimensional Groups and Borel Structures with Subalgebras of Ultraproduct, M.Sc. thesis, Shendi University, Sudan 2018.